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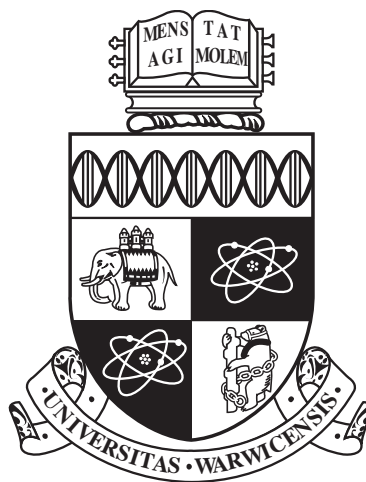
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**Splitting of separatrices in area-preserving maps
close to 1:3 resonance**

by

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Thesis

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Declarations

I, Giannis Moutsinas, declare that to the best of my knowledge this work is original work obtained in collaboration with my supervisor, Vassili Gelfreich, unless otherwise stated, cited, or commonly known

The material in this thesis has not to my knowledge been submitted for any other degree either at this university (the University of Warwick) or any other university. At the time of submission none of the work within this thesis has appeared or been submitted to any other publication

Abstract

We consider a real analytic family of area-preserving maps on \mathbb{C}^2 , f_μ , depending analytically on the parameter, such that f_0 is a map at 1:3 resonance. Such maps can be formally embedded in an one degree of freedom Hamiltonian system, called the normal form of the map. We denote the third iterate of the map by $F_\mu = f_\mu^3$.

We show that given a certain non degeneracy condition on the map F_0 , there exists a Stokes constant, θ , that when it does not vanish, it describes the splitting of the separatrices that the normal form predicts. We show that this constant can be approximated numerically for any non-degenerate map F_0 .

For a non-vanishing and small enough μ , we show that if the Stokes constant does not vanish the separatrices split. Moreover, let Ω be the area of the parallelogram defined by the 2 vectors tangent at the two separatrices at a homoclinic point. For any $M \in \mathbb{N}$ we have the estimate

$$\Omega(\mu) = \left(\sum_{n=0}^M \vartheta_n (\log \lambda_\mu)^n + O((\log \lambda_\mu)^{M+1}) \right) e^{-\frac{2\pi^2}{\log \lambda_\mu}}.$$

In this equation λ_μ is the largest eigenvalue of the saddle points around the origin and ϑ_n 's are real constants with $\vartheta_0 = 4\pi|\theta|$.

Chapter 1

Introduction

One of the fundamental questions of Hamiltonian systems is the one about the stability of periodic orbits. One way to answer this question is the first return map or Poincaré map. This map is constructed as the intersection of a periodic orbit in the state space of a Hamiltonian system with a certain lower-dimensional subspace transversal to the flow of the system.

As an example, we consider a 2 degrees of freedom Hamiltonian system. The state space of such a system is of dimension 4, but since we know that the Hamiltonian function is an integral of the system, by choosing a value for this function we can look at the surface that this defines and this drops the dimension to 3. Then we assume there exists a periodic orbit on this 3 dimensional surface. We choose a point of this periodic orbit and we consider a plane transversal to the orbit.

In order to construct the first return map we choose a point on the plane and we let the flow evolve until its trajectory crosses the plane again. Then we define the map such that the image of the point we chose under the map is the point of the first crossing. Notice that the intersection of the periodic trajectory and the plane is a fixed point of the map and usually it is considered to be the origin of the plane. This procedure is shown in Figure 1.1. This map, now defined on a neighbourhood of the origin on the plane, can be shown to preserve area, see [Arn90]. Formally we have the following definition.

Definition 1.1. Let $V \subset \mathbb{R}^2$ open with $0 \in V$ and let $f : V \rightarrow \mathbb{R}^2$ be a function analytic in V , such that $f(0) = 0$. If $\det f'(x) = 1$ for all $x \in V$, then we say that f is an area-preserving map of the plane.

If λ_1 and λ_2 are the eigenvalues of $f'(x)$, then $\lambda_1 \cdot \lambda_2 = 1$. For the stability of the

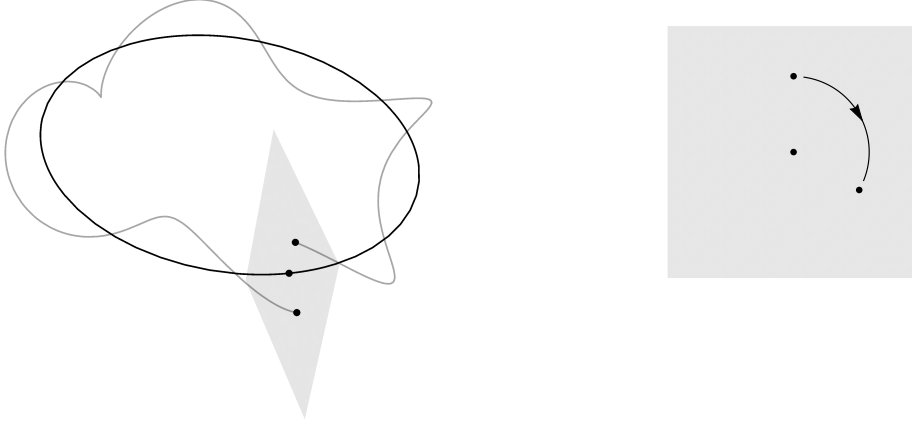


Figure 1.1: The construction of the first return map.

fixed point we have the following cases:

- $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 \neq \lambda_2$ called *hyperbolic fixed point*,
- $\lambda_{1,2} = e^{\pm i2\pi\rho}, \rho \in (0, \frac{1}{2}) \setminus \mathbb{Q}$ called *non-resonant elliptic fixed point*,
- $\lambda_{1,2} = e^{\pm i2\pi\rho}, \rho \in (0, \frac{1}{2}) \cap \mathbb{Q}$ called *resonant elliptic fixed point*,
- $\lambda_{1,2} = \pm 1$ called *parabolic fixed point*.

In a neighbourhood of a hyperbolic fixed point the map is the time-1 flow of a Hamiltonian system around a saddle. In a neighbourhood of a non-resonant elliptic fixed point the map is approximately a rotation, in general with a non-constant angle. The other two cases have many subcases. We will consider the case where $\lambda_{1,2} = e^{\pm i2\pi/3}$. An elliptic point with these eigenvalues is called an elliptic point at 1 : 3 resonance.

At first glance, resonant elliptic points seem rather improbable since they have codimension 1. However in order to construct the map we fix the value of the Hamiltonian. Naturally we can change this value and by the implicit function theorem we get the existence of a periodic trajectory in nearby values. So the value of the Hamiltonian is a “natural” unfolding parameter of the map which implies that in Hamiltonian systems resonant orbits appear generically.

1.1 Non-integrability of Hamiltonian systems

The phenomenon we will study here is connected with the non-integrability of Hamiltonian systems and was first observed by the French mathematician Henri Poincaré around 1890 when investigating the stability of the solar system. Poincaré considered the system formed by three bodies: Sun, Earth and Moon, under the action of Newton's laws of gravity. In an attempt to prove the stability of the three body system, he used perturbation series and realized its divergent character due to the presence of a transverse homoclinic orbit [Poi90]. He also noticed that a small differences in the initial positions or velocities of one of the bodies would lead to a radically different state when compared to the unperturbed system, what is now commonly known as deterministic chaos. Poincaré realized that a small perturbation can destroy a homoclinic connection and its place is taken by a region where the stable and the unstable manifolds intersect in a highly non-trivial way. He was even able to prove for a concrete example that the width of this region was exponentially small with respect to the size of the perturbation.

This splitting of separatrices is exactly the phenomenon we are interested here. We will give some brief historical remarks and we encourage the reader to see the survey by Gelfreich and Lazutkin [GL01] for a more detailed exposition of the theory until 2000.

Splitting of separatrices in area-preserving maps

The obvious way to address the above question of stability of periodic orbits in two degrees of freedom systems is to look directly at the map.

The first map to be treated was the Chirikov standard map, defined on the torus by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \varepsilon \sin(x) \\ y + \varepsilon \sin(x) \end{pmatrix}.$$

For $\varepsilon = 0$ the standard map is integrable but for $\varepsilon > 0$ the homoclinic separatrix splits. An asymptotic formula for the splitting in the standard map was published by Lazutkin in 1984 in a pioneering article, see [Laz03] for the english translation. However the proof was incomplete and it was completed and published by Gelfreich in [Gel99]. The same asymptotic formula was derived by Hakim and Mallick,

[HM93], using Ecalle's theory of resurgent functions. However their work was purely formal without rigorous proof.

Neishtadt, [Nei84], proved that the splitting in the difference of the two separatrices of analytic maps close to identity admits an exponentially small upper bound. Later Fontich and Simó, [FS90], using Lazutkin's methods gave a sharp upper bound.

Using the theory of resurgent functions, Gelfreich and Sauzin proved for an instance of the Hénon map at 1:1 resonance that the splitting of separatrices is exponentially small and provided the first asymptotic term for it, see [GS01].

More recently Martín, Sauzin and Seara have studied the splitting of separatrices in perturbations of the McMillan map, see [MSS11a] and [MSS11b]. Their approach combined the theory of resurgent functions with Lazutkin's original ideas.

A paper, [Gel02], was published by Gelfreich stating the first asymptotic term for the resonances 1:1, 1:2 and 1:3. However the only proof on these results published until now is a preprint by Brännström and Gelfreich [BG08]. There the authors derive and prove the asymptotic formula for area-preserving maps near a Hamiltonian saddle-centre bifurcation.

Non-autonomous perturbation of flows

An other way to address the above question is to embed the first return map into the flow of a non-autonomous Hamiltonian system of one degree of freedom. This enables the usage of methods developed for differential equations and more results are available. This flow can be written as a periodic time dependent perturbation of an one degree of freedom Hamiltonian system. More precisely, we describe this system with the help of the Hamiltonian function

$$H(\mu, \varepsilon; x, y, t) = H_0(x, y) + \mu H_1(x, y, \frac{t}{\varepsilon}, \varepsilon, \mu),$$

with $H_1(x, y, t, \varepsilon, \mu)$ periodic in t .

A natural question in this setting is whether homoclinic or heteroclinic connections that exists in the unperturbed system persist in the perturbed one.

The case where only μ is considered to be a small parameter was solved using the so called Melnikov method, see [Mel63]. In this case we can reparametrize time such that $\varepsilon = 1$. Then for the separatrices of the system it holds

$$W_{\mu, \varepsilon}^{\pm}(t_0, t) = W_0(t - t_0) + \mu W_1^{\pm}(t_0, t) + O(\mu^2), \quad \pm t \in [t_0, \infty).$$

We define the *Melnikov function* by

$$M(t_0) := \int_{-\infty}^{\infty} \{H_0, H_1\}|_{W_0(t-t_0), t} dt,$$

where $\{H, G\}$ is the Poisson bracket of H and G . For the difference between the two separatrices at t_0 , measured in a coordinate system that uses H_0 as the first of its coordinates, we get that the difference in the first component is

$$d(t_0) = \mu M(t_0) + O(\mu^2).$$

However, when both μ and ε are considered small, then ε cannot be ignored. These systems are called *rapidly forced* systems since the period of the perturbation becomes arbitrarily small.

Nekhoroshev, [Nek77], showed that in many degrees of freedom Hamiltonian systems, the phase space can be covered by domains where the system behaves as if it was integrable for some time. He showed that this time is exponentially large with the size of the perturbation. Neishtadt showed in [Nei84] that d actually admits an upper bound that is exponentially small with ε . Neishtadt's results were refined by Treshchev in [Tre97]. Fontich based on Lazutkin's ideas, [Fon95], showed that the exponent depends on the location of the singularities in the parameter of the unperturbed separatrix.

In rapidly forced systems the Melnikov function can become exponentially small with ε , but since the error term is polynomially small in μ , the error can become bigger than the approximation. This situation can be avoided of course when μ is a function of ε which decreases exponentially as ε goes to 0. Then the error is also exponentially small and the Melnikov method can still be applied. It was shown by Gelfreich in [Gel97b] that this can be relaxed to a polynomial dependence, $|\mu| \leq C\varepsilon^p$, with p big enough.

Stronger results have been proved in specific systems. Poincaré [Poi93] discovered the phenomenon of splitting by looking at the system described by the Hamiltonian

$$\frac{y^2}{2} + \cos x + a \sin x \cos \frac{t}{\varepsilon}.$$

He proved that in this system the splitting is exponentially small and he derived the first term of the asymptotic expansion. Poincaré's arguments require a to be exponentially small in ε and his result is the same that the Melnikov method provides. However, for an ε -independent a Melnikov's method provides a wrong estimate.

Treshchev [Tre96] and Gelfreich [Gel97a] independently showed that by obtaining a different asymptotic formula using the averaging method with a continuous parameter.

The most studied system has been the rapidly perturbed pendulum with a perturbation only depending on time,

$$\ddot{x} = \sin x + \mu \varepsilon^\eta \sin \frac{t}{\varepsilon}.$$

Many authors have published on this, gradually strengthening the result, see [HMS88], [Sch89], [DS92], [Ang93], [EKS93], [Gel94] and [Swa96].

Recently Gaivão and Gelfreich [GG11] used the generalized Swift-Hohenberg equation as an example to show the transversality of the homoclinic solutions near a Hamiltonian-Hopf bifurcation.

Baldoma, Fontich, Guardia and Seara [BFGS12] showed that in systems where $H_0 = \frac{y^2}{2} + V(x)$ with V an algebraic or trigonometric polynomial and $|\mu| \leq C\varepsilon^\eta$, the Melnikov method can be applied if $\eta > 0$. Moreover, they also showed that the Melnikov method fails when p becomes zero and they derived the first term of the asymptotic series in this case.

Splitting of separatrices in physics

The same phenomenon has been studied in physics although in a different framework. The common technique there is truncating an asymptotic series in the optimal order and then showing that the remainder is exponentially small. This technique is called asymptotics beyond all order or superasymptotics, see [Ber91], [STL12] or [IL05].

There exist many examples of problems for which asymptotic power series methods lead to divergent series. Oppenheimer [Opp28] while investigating a phenomenon in quantum physics known as the Stark effect, demonstrated that the lifetime of a certain quantum state was inversely proportional to a quantity exponentially small with the strength of the electric field applied at the system.

Kruskal and Segur [KS91] demonstrated that the geometric model for dendritic crystal growth fails to produce needle crystal solutions due to exponentially small effects, a byproduct of the breakage of a heteroclinic connection. This work has influenced many others in the field and the same technique has been applied at the formal level to prove the non-persistence of homoclinic or heteroclinic solutions

to certain singularly perturbed systems. Examples of application of this method include surface tension and wave formation [GJ95], [YA97], [Tov00], [VdBK09]), crystal growth [CM05] and optics [CK09]. More information about applications of exponentially small splitting to mechanics, fluids and optics can be found in the survey of Champneys [Cha98].

In his book [Lom00], Lombardi puts the superasymptotics into rigorous arguments that can be used to solve many problems in exponentially small phenomena. He did that by reducing the problem to the study of certain oscillatory integrals which describe the exponentially small terms.

1.2 Measuring the splitting of separatrices

Until now we talked about the splitting of separatrices without defining concretely what it means. The reason for this is that there are a handful of different quantities that were used to measure it. Let us describe them.

- The *splitting angle*. Measuring the angle that the two separatrices create at their intersection is an intuitive idea, since it cannot vanish when they meet transversally. However there are a few disadvantages: computing the angle requires the definition of a Riemannian metric, it depends on the homoclinic point chosen, and finally a symplectic change of variables changes also the angle.
- The *splitting amplitude* is defined as $\max_{t \in \{t_0, t_0+1\}} |W_\varepsilon^+(t) - W_\varepsilon^-(t)|$, where t_0 corresponds to a homoclinic point, $W_\varepsilon^+(t_0) = W_\varepsilon^-(t_0)$. However the splitting amplitude has the same disadvantages as the splitting angle.
- The *homoclinic invariant*, Ω , was introduced by Lazutkin in [GLS94]. It is defined by $\Omega = \omega(\dot{W}_\varepsilon^+(t_0), \dot{W}_\varepsilon^-(t_0))$ and it represents the area of the parallelogram formed by the tangent vectors to the separatrices at a point of intersection. The homoclinic invariant has the same value on all homoclinic points and it is invariant under canonical changes of coordinates.
- The *area of a crescent*. We choose a homoclinic point p_h and we look at the segments of the separatrices bounded by p_h and its image under the map $F_\varepsilon(p_h)$. If the separatrices split, there are finitely many points of intersection between these two segments. We choose two neighbouring ones and we measure the area the two separatrices define. This area is invariant not only under the action of the map but also under canonical changes of coordinates.

- The *width of the instability region*. One can show using KAM theorem, that the closure of the separatrices is contained in a domain bounded by invariant curves, each of which is diffeomorphic to a circle. One can speak of the ‘last’ invariant curve bounding the so-called instability region. The splitting of separatrices can be characterized by the width or the area of this region. Note that the width is not an invariant.

The last quantity is harder to estimate than the preceding ones. The relationship between the splitting amplitude and the width of the instability region was established by Lazutkin [Laz90] for the standard map, and a generalization of this result was obtained by Treshchev [Tre98].

In the present analysis we will use Lazutkin’s homoclinic invariant to measure the splitting.

1.3 Normal form of maps close to 1:3 resonance

An interesting class of area-preserving maps is the maps tangent to identity. The definition is given below.

Definition 1.2. Let f be an area-preserving map of the plane, if $f(0) = 0$ and $f'(0)$ is the identity then we call f a tangent to identity map.

For any tangent to identity area-preserving map, there exists an one degree of freedom Hamiltonian system, such that the map can be formally embedded in its flow. This implies that such map can always be approximated with arbitrary precision by a flow.

Let V be a neighbourhood of the origin in \mathbb{C}^2 and \mathcal{I} a neighbourhood of the origin in \mathbb{R} . Let $f_\mu : V \rightarrow \mathbb{R}^2$ be a real-analytic, area-preserving map for all $\mu \in \mathcal{I}$. Moreover let $f_\mu(0) = 0$, $f'_0(0)$ have eigenvalues $e^\pm = e^{\pm 2\pi i/3}$ and f_μ is C^∞ in μ .

Theorem 1.3 (Birkhoff normal form). *There is a formal canonical change of coordinates Φ such that the map $N = \Phi \circ f_0 \circ \Phi^{-1}$ commutes with the rotation $R_{2\pi/3}$, i.e.: $N \circ R_{2\pi/3} = R_{2\pi/3} \circ N$.*

The map N is called the *Birkhoff normal form* of f_0 and the map $R_{-2\pi/3} \circ N$ is tangent to identity. Since a tangent to identity map can be formally represented as the time-one flow of an autonomous Hamiltonian system, there is a formal Hamiltonian

H such that

$$N = R_{2\pi/3} \circ \phi_H^1,$$

where ϕ_H^1 is the time-one flow of H . The corresponding vector field is usually called *Takens normal form vector field*, see [Tak74]. The Hamiltonian inherits the symmetry of the normal form:

$$H \circ R_{2\pi/3} = H.$$

So H is a formal integral of N and by changing back to the original coordinates we get a formal integral of f_0 . Note that if the map f_0 is not integrable, H cannot be convergent. In the middle figure of Figure 1.2 the level lines of the third order of this Hamiltonian are shown.

The formal Hamiltonian H is not defined uniquely so there is room for further normalization.

Proposition 1.4 ([GG09]). *Let f_0 be as above. Then there is a formal Hamiltonian H and formal canonical change of variables which conjugates f_0 with $R_{2\pi/3} \circ \phi_H^1$. Moreover, H has the following form:*

$$H(x, y) = (x^2 + y^2)^3 A(x^2 + y^2) + (2x^3 - 6xy^2) B(x^2 + y^2), \quad (1.1)$$

where A and B are series in one variable with real coefficients:

$$A(I) = \sum_{\substack{k \geq 0 \\ k \not\equiv 2 \pmod{3}}} a_k I^k, \quad B(I) = \frac{b_0}{6} + \sum_{\substack{k \geq 1 \\ k \not\equiv 2 \pmod{3}}} b_k I^k$$

and the coefficient of A and B are uniquely defined if $b_0 \neq 0$.

For the coefficient b_0 it holds $b_0 = 6|h_{30}|$, where h_{30} is the 3rd order coefficient in Birkhoff normal form Hamiltonian.

For a map, f_μ , close to the resonance it holds:

Proposition 1.5 ([GG09]). *Let f_μ be as above and let the coefficient b_0 for the map f_0 not vanish. Then there is a formal Hamiltonian H and formal canonical change of variables which conjugates f_μ with $R_{2\pi/3} \circ \phi_H^1$. Moreover, H has the following form:*

$$H(\mu; x, y) = (x^2 + y^2) A(\mu, x^2 + y^2) + (2x^3 - 6xy^2) B(\mu, x^2 + y^2),$$

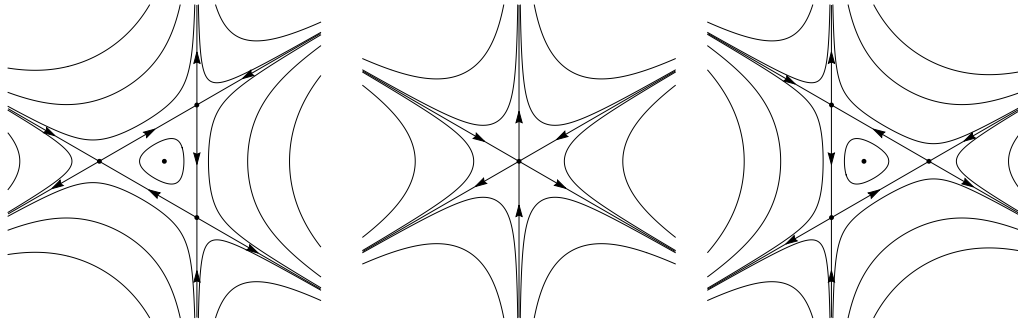


Figure 1.2: The normal form of the unfolded 1:3 resonant map.

where A and B are series in two variables with real coefficients:

$$A(\mu, I) = \sum_{\substack{k, m \geq 0 \\ k \not\equiv 1 \pmod{3}}} a_{k, m} I^k \mu^m, \quad B(\mu, I) = \frac{b_{0,0}}{6} + \sum_{\substack{k, m \geq 1 \\ k \not\equiv 2 \pmod{3}}} b_{k, m} I^k \mu^m,$$

with $b_{0,0} = b_0$ and $a_{0,0} = a_{1,0} = 0$. Moreover the coefficients of these series are unique.

1.4 Splitting of separatrices

The normal form predicts that close to resonance there are heteroclinic connections between the three saddle points. However since the convergence of the normal form is not given, it is natural to ask whether this prediction is correct. In his classical book *Mathematical Methods of Classical Mechanics*, V.I. Arnol'd conjectures that this is actually not true.

Since the class of maps is bigger than the class of flows, he states that there is no reason to expect all maps to act like flows. One of the implications of this is that the heteroclinic connections are not actually present. He also argues that the difference between the two separatrices has to be exponentially small since the normal form cannot detect it at any order and of course the presence of splitting implies that the normal form is divergent.

We will see that the splitting of the separatrices close to the 1:3 resonance is dominated by the splitting at exactly 1:3 resonance and that there is a transversal intersection in generic maps close to 1:3 resonance.

In order to simplify our analysis, we define $F_\mu := f_\mu^3$, i.e. the third iterate of the map f_μ . Then F'_0 has eigenvalues $\lambda_i = 1$, $i \in \{1, 2\}$. Since the normal form commutes

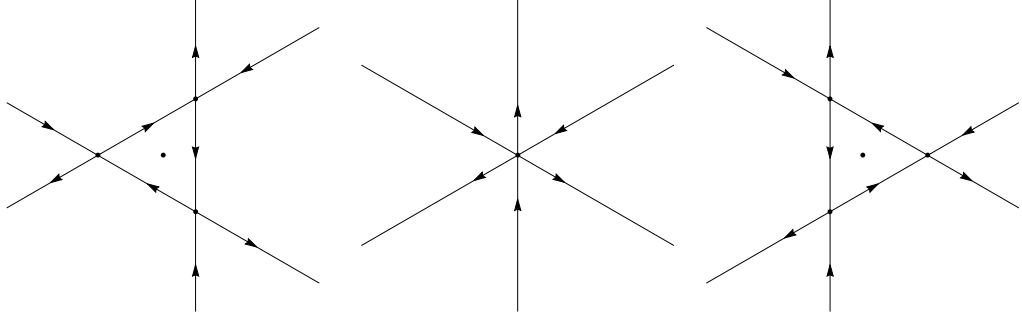


Figure 1.3: The separatrices of the normal form.

with the rotation by $2\pi/3$, the normal form Hamiltonian for F_μ is just the normal form Hamiltonian for f_μ multiplied by 3.

In Figure 1.3, the fixed points with the separatrices are shown. Notice that the separatrices of the flow are not invariant sets for f_μ , since f_μ maps one to the other, but are invariant sets for F_μ . From now on we will consider the map F_μ .

We see that at the resonance the stable and the instable separatrices of the origin do not meet at all so of course they do not split. To see the splitting at the resonance we need to study the map in a complex neighbourhood of the origin. This is done trivially since the map is considered to be analytic around the origin.

Map at resonance

We consider the vertical set of separatrices at resonance. Both of them are curves of dimension 1 in \mathbb{R}^2 , so when we complexify the map they become curves of complex dimension 1 in \mathbb{C}^2 . So we can draw the dynamics close to a separatrix by taking the projection on one of the coordinates. In Figure 1.4 we see what happens in the case of an integrable map. In this case the normal form is convergent and the map is just the time-1 flow of a Hamiltonian. This means that the invariant lines of the unstable and the stable separatrices coincide and all but the points on the real line have the fixed point as alpha and omega limit set.

On the other hand, when we consider a non-integrable map, we see that close to the real separatrix the dynamics is similar to the integrable case but as we move away the invariant lines start to oscillate. We see in Figure 1.5 that when the two separatrices are drawn together the splitting is apparent. It should be noted here that the splitting of the separatrices does not happen only on the plane that they are projected on. They also differ in the 2 dimensions that are perpendicular to the

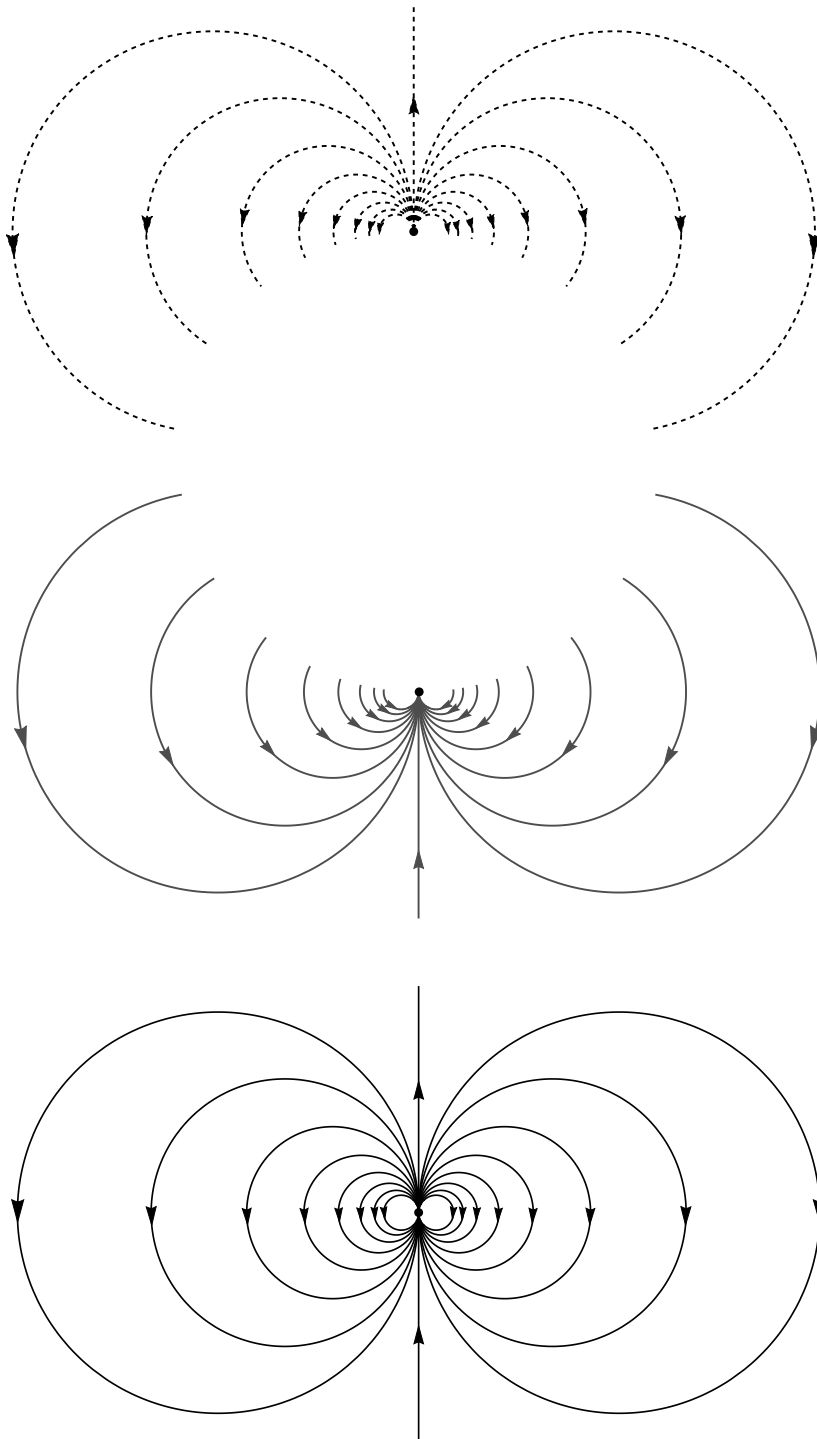


Figure 1.4: In the top figure the dynamics on the unstable separatrix of an integrable map at resonance are shown. In the middle figure the dynamics of the same map on the stable separatrix is shown and in the bottom picture the two pictures are combined.

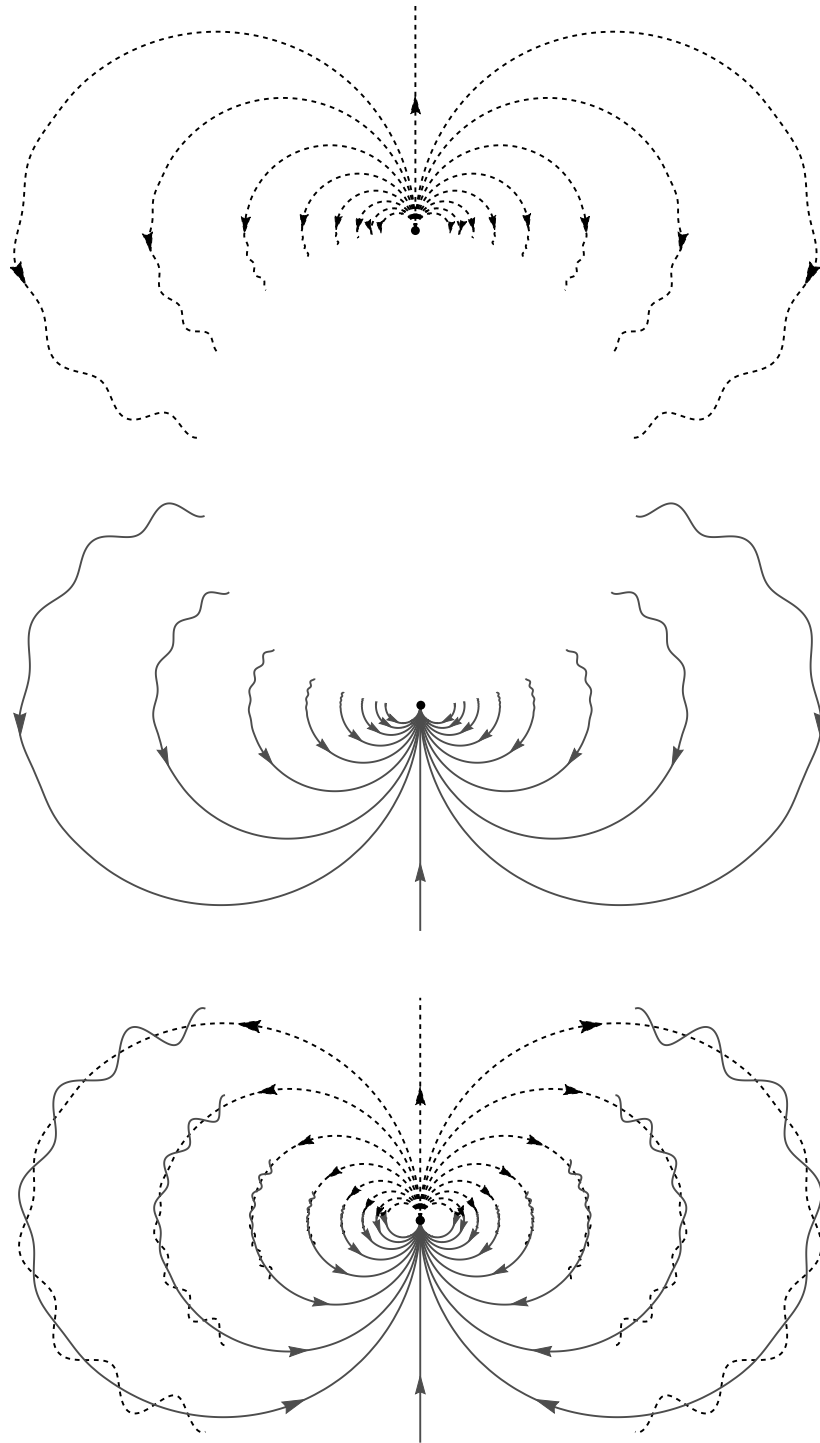


Figure 1.5: In the top figure the dynamics on the unstable separatrix of a non-integrable map at resonance are shown. In the middle figure the dynamics of the same map on the stable separatrix is shown and in the bottom picture the two pictures are combined.

plane of projection.

Map close to resonance

As we saw, for a map close to resonance two saddle points appear on the vertical set of separatrices. The separatrices are again curves of dimension 1 in \mathbb{R}^2 and curves of complex dimension 1 in \mathbb{C}^2 when the complexified map is considered. As in the resonant case we can draw the invariant lines on the projection of the separatrix on the second coordinate. In Figure 1.6 the integrable case is shown. We see that again the stable and unstable invariant lines coincide and every point but a half-line have the unstable fixed point as alpha limit set. Similarly, every point but a half-line have the stable fixed point as omega limit set.

In the non-integrable case an oscillation appears in both separatrices away from the fixed points, see Figure 1.7. Comparing the Figures 1.5 and 1.7 we see that even though at a neighbourhood of the resonant fixed point the change is dramatic, away from it the dynamics do not change a lot.

The separatrices are analytic functions so of course their difference is also an analytic function. With analytic functions being global objects, it is reasonable to expect that the difference close to the fixed points can be calculated by the difference away from them. Then since the dynamics away from the fixed points do not change significantly with the unfolding, it is reasonable to expect that the difference at the resonance dominates the difference of the unfolding. We will show that is actually the case.

1.5 Results

Once a non-integrable map gets unfolded the splitting of the separatrices appear on \mathbb{R}^2 . As shown in Figure 1.8 the heteroclinic connections get destroyed and separatrices meet transversally in a complicated way. In order to measure this splitting we will use the homoclinic invariant¹ Ω .

In this section we summarize the results of this thesis. In Chapter 3 we deal with the map at resonance. In Chapter 4 we derive the asymptotic formula for the homoclinic

¹ One could argue that the word that should be used here is heteroclinic instead of homoclinic. There are basically two reasons for this choice. One is historical, since this is the name originally used. The second is that this connection could actually be viewed as a homoclinic one. Recall that the hyperbolic points are fixed points of the map F_μ and not of the map f_μ . For the map f_μ these three points form a 3-periodic orbit, so their separatrices are separatrices of this periodic orbit.

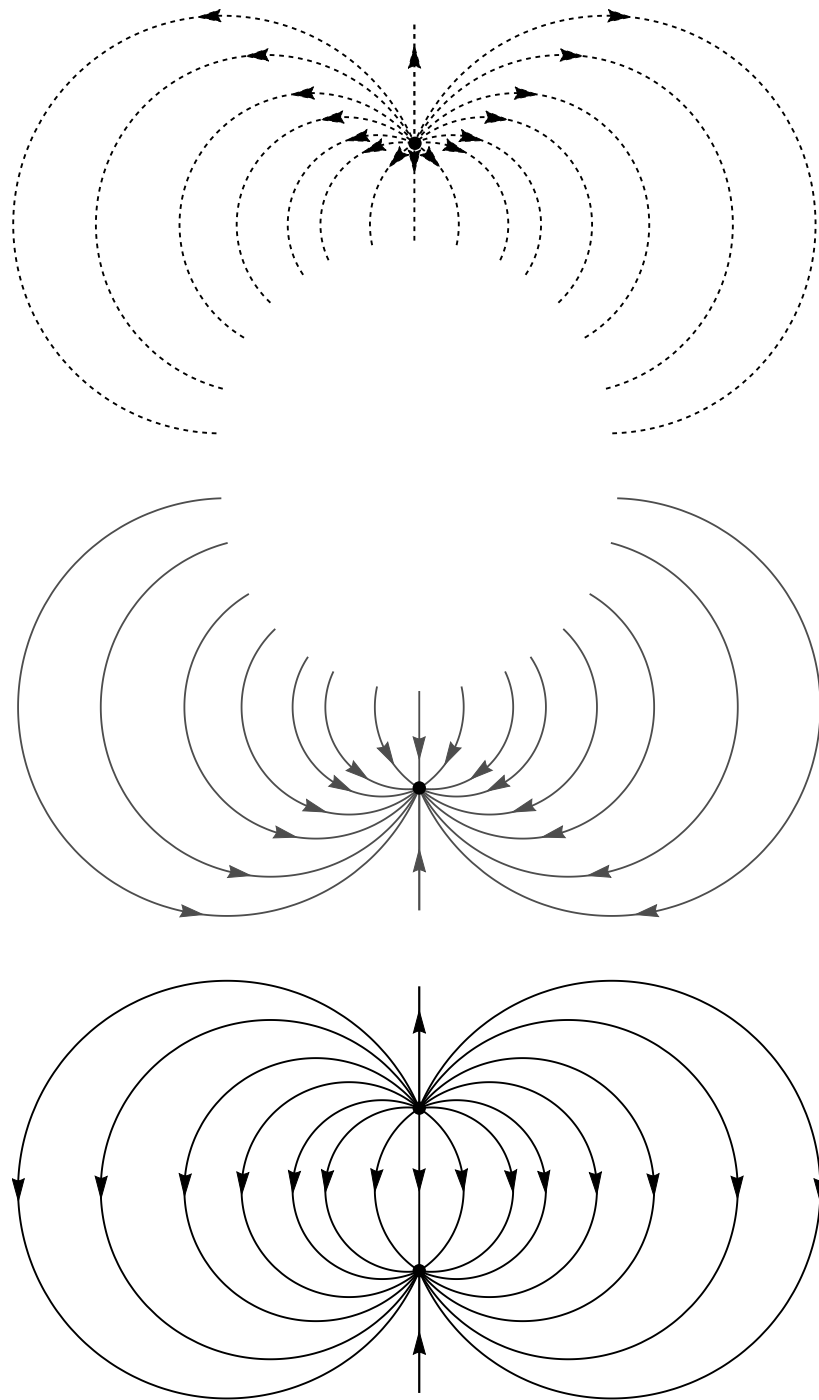


Figure 1.6: In the top figure the dynamics on the unstable separatrix of an integrable map close to resonance are shown. In the middle figure the dynamics of the same map on the stable separatrix is shown and in the bottom picture the two pictures are combined.

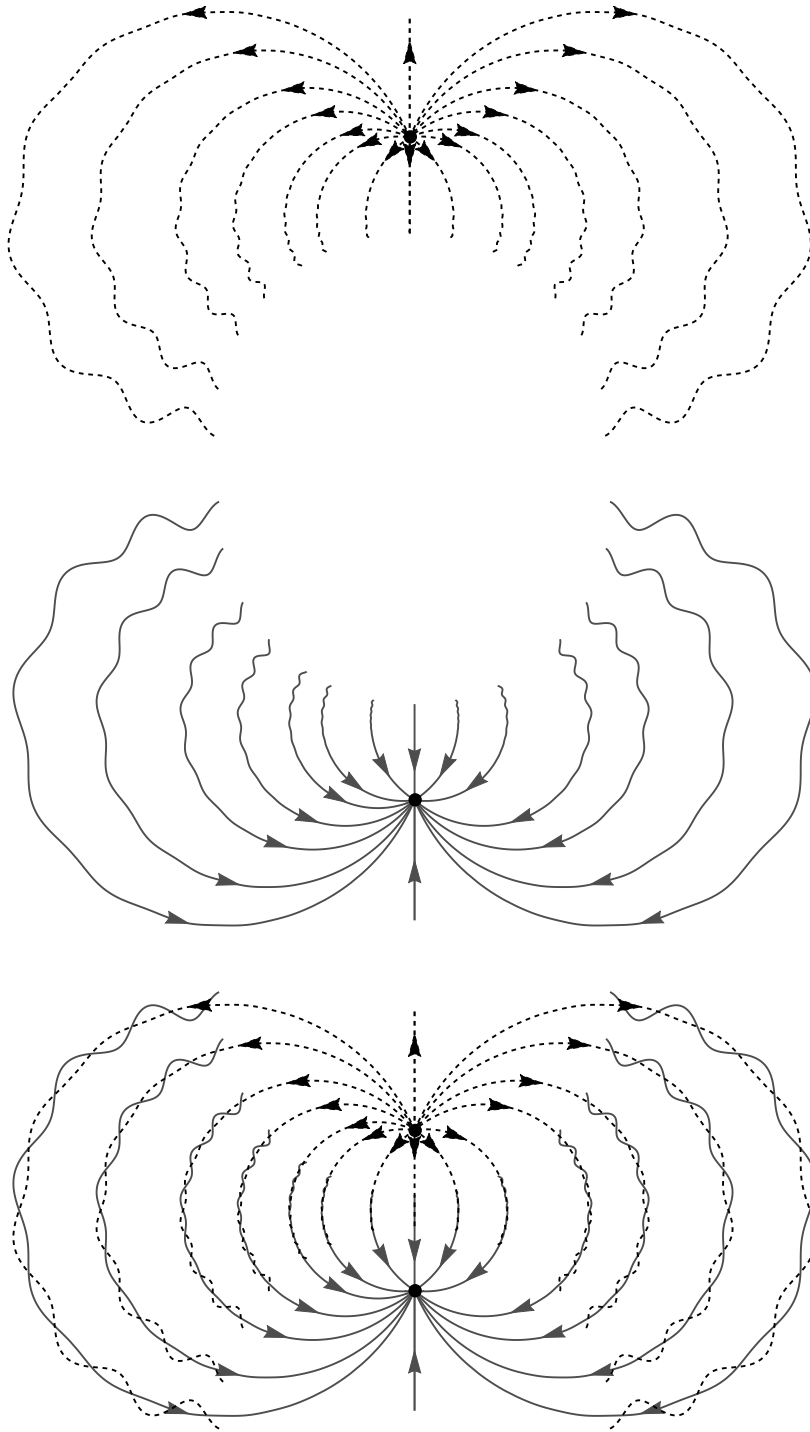


Figure 1.7: In the top figure the dynamics on the unstable separatrix of a non-integrable map close to resonance are shown. In the middle figure the dynamics of the same map on the stable separatrix is shown and in the bottom picture the two pictures are combined.

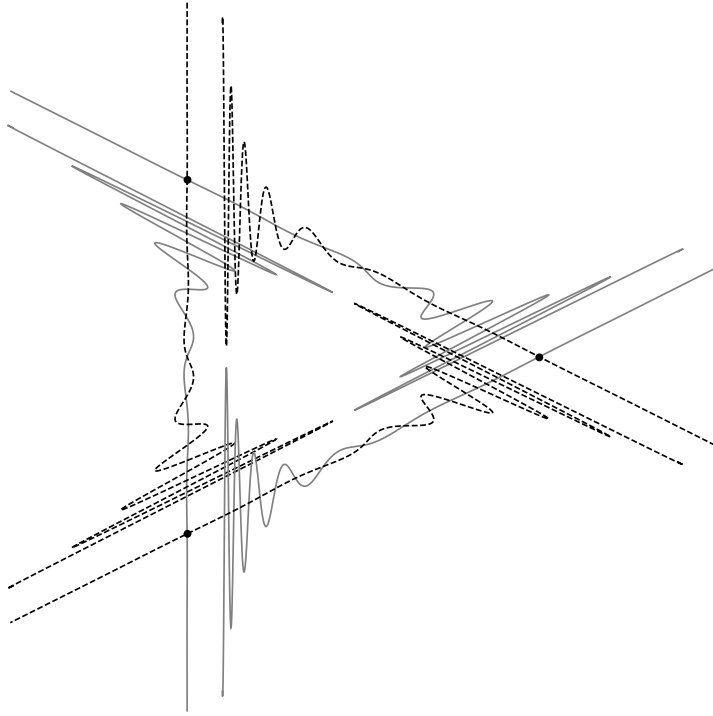


Figure 1.8: The splitting of the triangle that separatrices form.

invariant. Finally in Chapter 5 we provide a numerical method to compute the Stokes constant of a resonant map.

The theorems of the following chapters are repeated here however the wording has been slightly changed to avoid referring to notions that have not been defined yet. The reader should treat this section just as a summary of the results and is advised to refer to the respective chapters for any other purpose.

Assumptions on the map

Let V be a neighbourhood of the origin in \mathbb{C}^2 and \mathcal{I} a neighbourhood of the origin in \mathbb{R} . Let $f_\mu : V \rightarrow \mathbb{R}^2$ be a real-analytic, area-preserving map for all $\mu \in \mathcal{I}$. Let moreover $f_\mu(0) = 0$, $f'_0(0)$ have eigenvalues $e^\pm = e^{\pm 2\pi i/3}$ and f_μ be analytic around 0 in μ . We assume that the coefficient $b_{0,0}$ of the normal form does not vanish. We define $F_\mu = f_\mu^3$. From now on F_μ will denote the third iterate of an area-preserving map close to 1:3 resonance.

Chapter 3 results

Let F_0 agrees at least up to order 4 with the normal form of Proposition 1.4. We consider the equation

$$W(t+1) = F_0(W(t)). \quad (1.2)$$

Theorem 3.1. *There exists a unique formal solution with real coefficients,*

$$\mathcal{W}(t) = \begin{pmatrix} 0 \\ -\frac{1}{b_0 t} \end{pmatrix} + O(|t|^{-3}) \in \frac{1}{t} \mathbb{C}[[\frac{1}{t}]]^2,$$

of equation (1.2) and any other formal solution of the form $\mathcal{W}'(t) = (0, -\frac{1}{b_0 t}) + O(|t|^{-2})$ can be written as $\mathcal{W}(t+c)$ for some $c \in \mathbb{C}$. Moreover there exists a formal solution with real coefficients, $\tilde{\Xi} \in t^2 \mathbb{C}[[\frac{1}{t}]]^2$, of the equation

$$X(t+1) = F'_0(\mathcal{W}(t)) \cdot X(t),$$

such that

$$\tilde{\Xi}(t) = \begin{pmatrix} b_0 t^2 - 18 \frac{b_1}{b_0^2} + \frac{24b_1^2}{b_0^3} t^{-2} \\ -\frac{8a_0}{b_0^3} t^{-1} \end{pmatrix} + O(|t|^{-3}),$$

and $\det(\tilde{\Xi}(t), \dot{\mathcal{W}}(t)) = 1$.

The Borel transform of \mathcal{W} is a function, \hat{W} , analytic around the origin with singularities at $2\pi i \mathbb{Z}_*$ and is of exponential type along any path that crosses the imaginary axis finitely many time and does not go to infinity vertically.

The Borel-Laplace summation of \mathcal{W} gives two solutions of the equation (1.2), W^+ and W^- , that satisfy $\lim_{t \rightarrow \pm\infty} W^\pm(t) = 0$. There exist two complex constants, θ and ρ , such that for any $t \in \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq 1, \operatorname{Im}(z) < 0\}$, with $|t|$ big enough, it holds

$$W^+(t) - W^-(t) \asymp e^{-2\pi i t} \left(\theta \tilde{\Xi}(t) + \rho \dot{\mathcal{W}}(t) \right) + O(t^7 e^{-4\pi i t})$$

and

$$\theta = \lim_{t \rightarrow +\infty} e^{2\pi t} \omega(W^+(-it) - W^-(-it), \dot{W}^-(-it)).$$

The constant θ will be called the *Stokes constant* of the map F_0 and it gives the size of the splitting on the transversal direction.

Chapter 4 results

Notice that for the next theorem we assume the map is as described above but also we have an extra assumption that the Stokes constant of the resonant map does not vanish.

Theorem 4.2. *Let F_μ be an area-preserving map that agrees with the normal form stated in section 1.3 up to degree 4 and that F_0 is the third iterate of non-degenerate area-preserving map at resonance 1:3. For $\mu \neq 0$, let λ_μ denote the largest eigenvalue of its saddle points and let Ω be the Lazutkin homoclinic invariant of the map. If the Stokes constant θ of the resonant map does not vanish, then there exist $\mu_0 > 0$ and real constants ϑ_n such that for any $\mu \in (-\mu_0, \mu_0) \setminus \{0\}$ and any $M \in \mathbb{N}$ it holds*

$$\Omega(\mu) = \left(\sum_{n=0}^M \vartheta_n (\log \lambda_\mu)^n + O((\log \lambda_\mu)^{M+1}) \right) e^{-\frac{2\pi^2}{\log \lambda_\mu}}.$$

Moreover $\vartheta_0 = 4\pi|\theta|$.

Chapter 5 results

Let \mathscr{W}_N denote the truncation of the formal series \mathscr{W} to order N .

Theorem 5.3. *For $M, N \in \mathbb{N}$, $M, N \geq 2$, there exists $t_0 \geq 1$ such that for $w_N(t) = \mathscr{W}_N(t) + O(|t|^{-N-1})$, $w_M(t) = \mathscr{W}_M(t) + O(|t|^{-M-1})$ and all $t \in \{\mathbb{C} : |t| > t_0, \operatorname{Re}(t) \leq 0\}$, the following are true.*

1. *The limit $W^-(t) := \lim_{m \rightarrow \infty} F_0^m(w_N(t-m))$ exists, is an analytic function and*

$$\lim_{m \rightarrow \infty} F_0^m(w_M(t-m)) = \lim_{n \rightarrow \infty} F_0^n(w_N(t-n)).$$

2. $W^-(t) = F_0(W^-(t-1)).$

3. *There exists $C_1 > 0$ such that $\|W^-(t) - w_N(t)\|_\infty \leq C_1|t|^{-N-1}$.*

4. *There exists $C_2 > 0$ such that for all $m \in \mathbb{N}$*

$$\|W^-(t) - F_0^m(w_N(t-m))\|_\infty \leq C_2|t-m|^{-N+1}.$$

5. *There exists $C_3 > 0$ such that for all $m \in \mathbb{N}$*

$$\|\dot{W}^-(t) - (F_0^m)'(w_N(t-m)) \cdot \dot{w}_N(t-m)\|_\infty \leq C_3|t-m|^{-N}.$$

We use this theorem to assess the expected error of numerical experiments and we show numerically that for an instance of the Hénon map,

$$H : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto R_{2\pi/3} \cdot \begin{pmatrix} x \\ y - x^2 \end{pmatrix},$$

the Stokes constant does not vanish.

Chapter 2

Preliminaries

In this chapter the main ideas behind resurgence will be presented. Most of the results stated here will not be proved, the reader is referred to the bibliography for the proofs. The theory originated from the work of Écalle [Éca81]. Unfortunately his books are not available in English. An introduction in English can be found in [Sau08, Sau13a, SS96]. In this exposition we will loosely follow [Sau08] and [Sau13a]. In [SS96] the definition of resurgent functions is aimed to be used in ODEs and PDEs and it is slightly more restrictive but also slightly stronger. A proof of existence of symmetric paths can be found in [Sau13b]. The content of sections 2.5.7 and 2.5.8 is original work unless it is specified otherwise.

2.1 Notation

We denote by \mathbb{N} the set of all positive integers, the same set with 0 added will be denoted by \mathbb{N}_0 . By \mathbb{R} and \mathbb{C} we denote the sets of real and complex numbers respectively. By \mathbb{R}^+ we denote the set of all positive reals and \mathbb{R}_0^+ is the set of all non negative reals. Similarly we define \mathbb{R}^- and \mathbb{R}_0^- . We denote by $\mathbb{D}_r(z)$ the open disk centered at z of radius r and by $\mathbb{D}_r^*(z)$ the same disk minus its center. We denote by ω the standard symplectic form on \mathbb{C}^2 .

Throughout this text we will use the pair of variables t and s as duals of each other. The Laplace transform will always be applied to a function of s and the Borel transform will always be applied to a function or a formal series of $\frac{1}{t}$. Moreover when we talk about the ring of formal or convergent series it will be implicitly assumed that this is a ring under multiplication if the variable is t and a ring under convolution if the variable is s .

We will abuse the notation s^n . This will have both the normal meaning of the number that s represents raised to the power of n , but also it will represent the function $s \mapsto s^n$. Finally $f(x)^2$ will denote the product $f(x) \cdot f(x)$ and $f^2(x)$ will denote $f \circ f(x)$. This notation extends to any integer.

2.2 Useful functional spaces

2.2.1 The space $C^\omega(\mathbb{C})$ of entire functions

Let $C^\omega(\mathbb{C})$ be the space of entire functions equipped with the topology generated by the family of seminorms

$$\|f\|_{\mathbb{D}_n} := \sup_{s \in \mathbb{D}_n} |f(s)|,$$

with $n \in \mathbb{N}$. Under this topology $C^\omega(\mathbb{C})$ becomes a Fréchet space.

2.2.2 The spaces $\mathcal{B}_n(\mathbb{D}_r)$

Let $r > 0$ and $g(s)$ be a function analytic on \mathbb{D}_r . We define $\|g\|_n := \sup_{s \in \mathbb{D}_r} |s^{-n} g(s)|$, $n \in \mathbb{N}$. This implies $|g(s)| \leq |s|^n \|g\|_n$. We define $\mathcal{B}_n(\mathbb{D}_r) := \{g \in C^\omega(\mathbb{D}_r) : \|g\|_n < \infty\}$.

It is trivial to check that $\|\cdot\|_n$ is a norm on $\mathcal{B}_n(\mathbb{D}_r)$. Let $\{g_n\}_{n \geq 0}$ be Cauchy in $\mathcal{B}_n(\mathbb{D}_r)$,

$$|g_n(s) - g_m(s)| \leq |s|^n \|g_n - g_m\|_n \leq r^n \|g_n - g_m\|_n.$$

So Cauchy in $\mathcal{B}_n(\mathbb{D}_r)$ implies uniform Cauchy in \mathbb{D}_r , which implies that the limit is in $\mathcal{B}_n(\mathbb{D}_r)$, thus $(\mathcal{B}_n(\mathbb{D}_r), \|\cdot\|_n)$ is Banach.

Evidently if $f \in \mathcal{B}_n(\mathbb{D}_r)$ then around the origin f is of the form $c s^n + O(s^{n+1})$. So if $f \in \mathcal{B}_{n+m}(\mathbb{D}_r)$, then $\|f\|_n \leq r^m \|f\|_{n+m}$. This implies that for $n, m > 0$ it holds $\mathcal{B}_{n+m}(\mathbb{D}_r) \subset \mathcal{B}_n(\mathbb{D}_r)$.

Notice that the space $\mathcal{B}_n(\mathbb{D}_r) \times \mathcal{B}_n(\mathbb{D}_r) \equiv \mathcal{B}_n(\mathbb{D}_r)^2$ with the norm $\|\cdot\|_{\times 2, n}$ defined by

$$\|(f, g)\|_{\times 2, n} := \max\{\|f\|_n, \|g\|_n\}.$$

is also Banach.

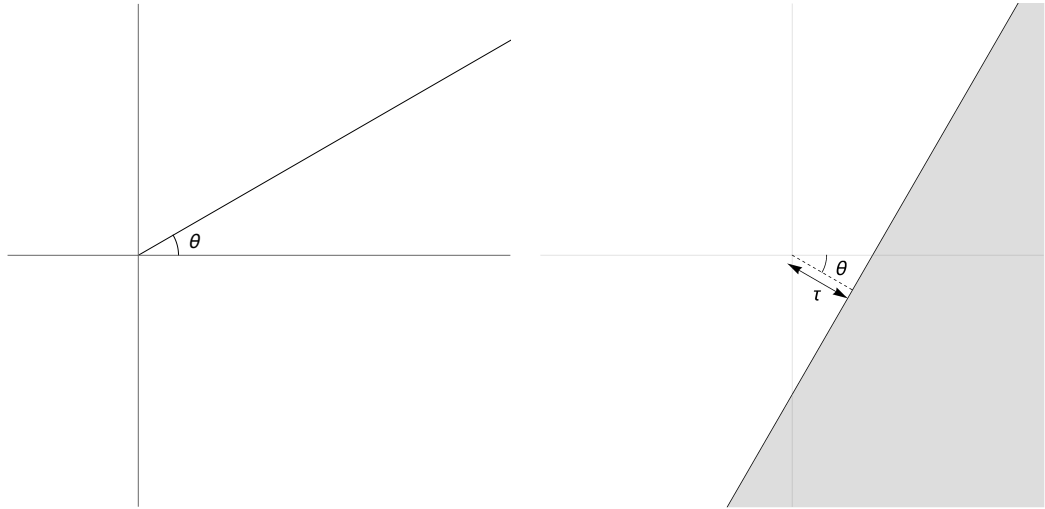


Figure 2.1: The Laplace transform of a function of exponential type τ in the direction θ gives rise to a function analytic in the half-plane $\{\operatorname{Re}(te^{i\theta}) > \tau\}$.

2.3 Classical Borel and Laplace transforms

In this section we present some elementary properties of the Laplace transform and of its formal inverse, the Borel transform. For an in depth treatment the reader should refer to one of the numerous books on the subject, we mention [Sch99] as an example. The Laplace transform is usually defined as an integral from 0 to ∞ . Here we will use a slightly more general definition.

Definition 2.1. Let $\theta \in \mathbb{R}$ and $\hat{\phi}$ be such that $r \mapsto \hat{\phi}(re^{i\theta})$ is analytic on a neighbourhood of \mathbb{R}^+ and $|\hat{\phi}(s)| \leq Ce^{\tau|s|}$. Functions with this property are called *functions of exponential type along the direction θ* . If the previous bound holds in every direction, we just say that the function is of exponential type. Moreover if τ' is the infimum of all such τ we say that the function is of *exponential type τ'* . We define the Laplace transform in the direction θ as the linear operator \mathcal{L}^θ ,

$$\mathcal{L}^\theta \hat{\phi}(t) := \int_0^{e^{i\theta}\infty} e^{-ts} \hat{\phi}(s) ds.$$

The function $\mathcal{L}^\theta \hat{\phi}$ is analytic on the half-plane $\operatorname{Re}(te^{i\theta}) > \tau'$, see Figure 2.1.

We define the convolution of two functions by

$$f * g(s) = \int_0^s f(t)g(s-t)dt.$$

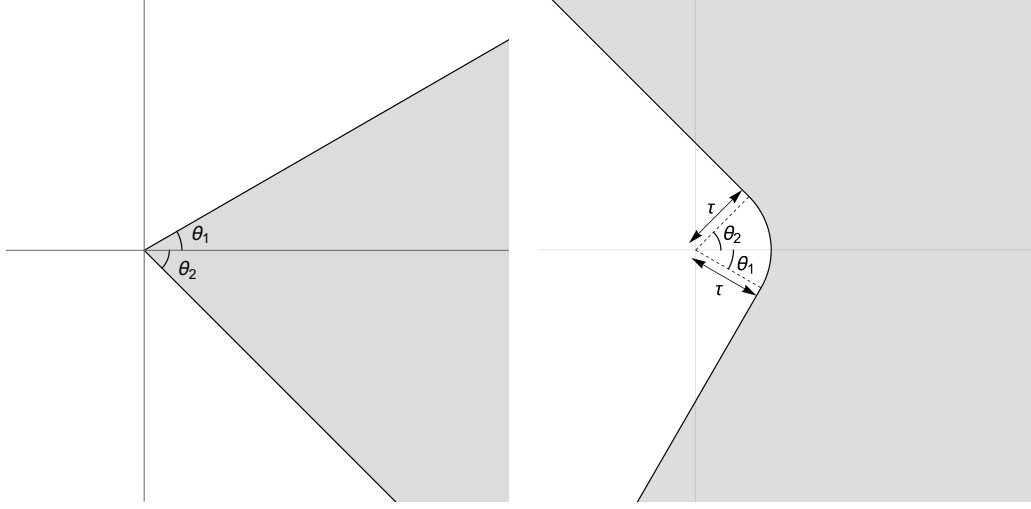


Figure 2.2: If $\hat{\phi}$ is analytic and of exponential type τ in a sector, then the domain of analyticity of $\mathcal{L}^{(\theta_1, \theta_2)} \hat{\phi}$ is the union of all possible half-planes.

The Laplace transform transforms convolution to multiplication. i.e.

$$\mathcal{L}^\theta[\hat{f} * \hat{f}] = \mathcal{L}^\theta[\hat{f}] \cdot \mathcal{L}^\theta[\hat{f}].$$

If the function $\hat{\phi}$ is analytic in a sector $\{s \in \mathbb{C} | \theta_1 < \arg s < \theta_2\}$, with $\theta_2 - \theta_1 < \pi$, and is of exponential type τ in that sector, then the Laplace transform converges on any θ in the sector and the function $\mathcal{L}^{\theta_1} \hat{\phi}$ is the analytic continuation of $\mathcal{L}^{\theta_2} \hat{\phi}$. So we can define the function $\mathcal{L}^{(\theta_1, \theta_2)} \hat{\phi}$ which is analytic in the union of the domains of analyticity of $\mathcal{L}^\theta \hat{\phi}$ for all θ in the sector, see Figure 2.2.

We can define $\mathcal{L}^{(\theta_1, \theta_2)} \hat{\phi}$ also when $\pi < \theta_2 - \theta_1 < 2\pi$. The situation is essentially the same with the only difference that the function $\mathcal{L}^{(\theta_1, \theta_2)} \hat{\phi}$ might be multivalued.

Let $\theta \in (0, \frac{\pi}{2})$. If $\hat{\phi}$ is of exponential type τ in the sectors $S_\theta = \{s \in \mathbb{C} | -\theta < \arg s < \theta\}$ and $S_{-\theta} = \{s \in \mathbb{C} | \pi - \theta < \arg s < \pi + \theta\}$ but it is not of exponential type in $\mathbb{C} \setminus (S_\theta) \cap S_{-\theta}$, then one can define $\mathcal{L}^{(-\theta, \theta)} \hat{\phi}$ and $\mathcal{L}^{(\pi - \theta, \pi + \theta)} \hat{\phi}$. See Figure 2.3. However at the points t where both are defined their difference cannot be identically equal to 0.

If $\hat{\phi}$ has a pole at finite distance from the origin, then its Taylor series has a positive but finite radius of convergence. This implies that the Laplace transform of its Taylor series, applied termwise, has 0 radius of convergence.

Let $\mathbb{C}[[s]]$ denote the space of formal power series of s with complex coefficients. We denote by $t^{-1}\mathbb{C}[[t^{-1}]]$ the space of formal series of negative powers of t without

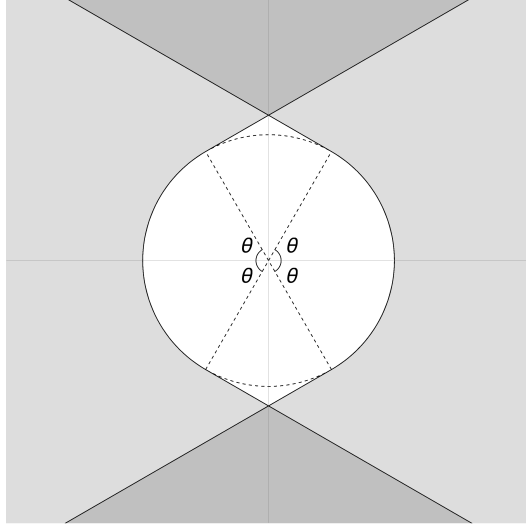


Figure 2.3: The domains of $\mathcal{L}^{(-\theta, \theta)} \hat{\phi}$ and $\mathcal{L}^{(\pi-\theta, \pi+\theta)} \hat{\phi}$.

constant term.

Because $\int_0^\infty \frac{s^n}{n!} e^{-ts} ds = t^{-n-1}$ for $\operatorname{Re} t > 0$, we have for any θ

$$\mathcal{L}^\theta \left[\frac{s^n}{n!} \right] (t) = t^{-n-1}, \quad \operatorname{Re}(te^{i\theta}) > 0.$$

Using this we define the formal Laplace transform $\mathcal{L}^\theta : \mathbb{C}[[s]] \rightarrow t^{-1}\mathbb{C}[[t^{-1}]]$.

Definition 2.2. The formal Borel transform is the linear operator

$$\mathcal{B} : \tilde{\phi}(t) = \sum_{n \geq 0} \frac{c_n}{t^{n+1}} \in t^{-1}\mathbb{C}[[t^{-1}]] \quad \mapsto \quad \hat{\phi}(s) = \sum_{n \geq 0} c_n \frac{s^n}{n!} \in \mathbb{C}[[s]].$$

Notice that the Borel transform is formally the inverse of the Laplace transform. This means that since the Laplace transform turns convolution into multiplication, the Borel transform turns multiplication into convolution.

If $\tilde{\phi}$ has a positive radius of convergence, if for example it converges for $t^{-1} < \rho$, then $\hat{\phi}$ defines an entire function of exponential type ρ^{-1} .

Let $\tilde{\phi} \in t^{-1}\mathbb{C}[[t^{-1}]]$ be divergent and let $\hat{\phi} = \mathcal{B}\tilde{\phi} \in \mathbb{C}[[s]]$ have a positive radius of convergence. Still it may happen that $\hat{\phi}$ extends analytically and is of exponential type in sectors. In these cases the Laplace transform converges in each sector but generally each sector defines a different function. This implies that we can possibly get a function from a formal series by taking the Laplace transform of the analytic continuation of the Borel transform of the series.

We denote by \mathcal{E} the analytic continuation of a function defined as a convergent power series around the origin. We define

$$\mathcal{I}^{(\theta_1, \theta_2)} = \mathcal{L}^{(\theta_1, \theta_2)} \circ \mathcal{E} \circ \mathcal{B}$$

to be the Borel Laplace summation over the sector (θ_1, θ_2) . Similarly if for some $\tilde{\phi} \in t^{-1}\mathbb{C}[[t^{-1}]]$, $\mathcal{I}^{(\theta_1, \theta_2)}[\tilde{\phi}]$ is a function analytic at a domain like the one in right figure of 2.2, we say that $\tilde{\phi}$ is Borel-Laplace summable.

The Borel Laplace summation is regular, i.e. it sends a convergent series to its function. It is linear and it commutes with multiplication, differentiation, integration, translation of the argument and composition. These imply that if $\tilde{\phi}$ satisfies formally some analytic equation then its Borel Laplace sum satisfies the same equation.

2.4 Formal series

2.4.1 The multiplicative ring $\mathbb{C}[[t^{-1}]]$

Recall that by $\mathbb{C}[[t^{-1}]]$ we denote the space of complex formal power series of t^{-1} . Addition, multiplication by a constant and multiplication of two series can be defined in a straightforward way. Let $\tilde{A}(t) = \sum_{n \geq 0} a_n t^{-n}$ and $\tilde{B}(t) = \sum_{n \geq 0} b_n t^{-n}$, $c \in \mathbb{C}$. Then

$$\begin{aligned} c\tilde{A}(t) &= \sum_{n \geq 0} c a_n t^{-n}, \\ \tilde{A}(t) + \tilde{B}(t) &= \sum_{n \geq 0} (a_n + b_n) t^{-n}, \\ \tilde{A}(t)\tilde{B}(t) &= \sum_{n \geq 0} \left(\sum_{m=0}^n a_m b_{n-m} \right) t^{-n}. \end{aligned}$$

Division $\tilde{A}(t)/\tilde{B}(t)$ is well defined if and only if $b_0 \neq 0$. These imply that $\mathbb{C}[[t^{-1}]]$ is a ring. Moreover the usual derivation $d_t := \frac{d}{dt}$ acts on $\mathbb{C}[[t^{-1}]]$, so $\mathbb{C}[[t^{-1}]]$ is a differential ring.

We define the *valuation* on $\mathbb{C}[[t^{-1}]]$ as the map $\text{val} : \mathbb{C}[[t^{-1}]] \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$\text{val}(\tilde{A}) := \min\{n \in \mathbb{N} | a_n \neq 0\}$$

and $\text{val}(0) := \infty$. With this we can define a metric on $\mathbb{C}[[t^{-1}]]$ by

$$\mu(\tilde{A}, \tilde{B}) := 2^{-\text{val}(\tilde{A} - \tilde{B})}.$$

Using this metric we can define a topology under which $\mathbb{C}[[t^{-1}]]$ is complete. In particular in this topology if a map from $\mathbb{C}[[t^{-1}]]$ to $\mathbb{C}[[t^{-1}]]$ is such that any given coefficient of the result depends on finitely many coefficients of the input then the map is continuous.

Example 2.3. We will see that multiplication in $\mathbb{C}[[t^{-1}]]$ is a continuous operation.

Let $\tilde{A}_N(t) = \sum_{n=0}^N a_n t^{-n}$. Then we have

$$\tilde{A}_N(t)\tilde{B}(t) = \sum_{n \geq 0} \left(\sum_{m=0}^n a_m \cdot b_{n-m} \cdot \mathbf{1}_{\{0, \dots, N\}}(m) \right) t^{-n},$$

where $\mathbf{1}_{\{0, \dots, N\}}$ is the indicator function of the integers from 0 to N . We see that for any $n \leq N$ the n -th coefficient of the product $\tilde{A}_N \cdot \tilde{B}$ agrees with the n -th coefficient of $\tilde{A} \cdot \tilde{B}$. This implies that

$$\mu(\tilde{A} \cdot \tilde{B}, \tilde{A}_N \cdot \tilde{B}) \leq 2^{-N-1},$$

so

$$\lim_{N \rightarrow \infty} \tilde{A}_N(t)\tilde{B}(t) = \tilde{A}(t)\tilde{B}(t).$$

2.4.2 The convolutive ring $\mathbb{C}[[s]]$

We defined $t^{-1}\mathbb{C}[[t^{-1}]]$ as the space of formal series without constant term, which is a maximal ideal in $\mathbb{C}[[t^{-1}]]$. The formal Borel transform maps $t^{-1}\mathbb{C}[[t^{-1}]]$ into $\mathbb{C}[[s]]$.

The space $\mathbb{C}[[s]]$ has a similar structure as $\mathbb{C}[[t^{-1}]]$ but instead of considering it as a ring with multiplication, we consider it as a ring with convolution. More precisely we define

$$\mathbb{C}[[s]] = \left\{ \sum_{n \geq 0} a_n \frac{s^n}{n!} \mid a_n \in \mathbb{C}, \forall n \in \mathbb{N} \right\}$$

and by the usual definition of convolution we get

$$\frac{s^n}{n!} * \frac{s^m}{m!} = \frac{s^{n+m+1}}{(n+m+1)!}.$$

So for $\hat{A}, \hat{B} \in \mathbb{C}[[s]]$ we get

$$\hat{A} * \hat{B}(s) = \sum_{n \geq 0} \left(\sum_{m=0}^n a_m b_{n-m} \right) \frac{s^{n+1}}{(n+1)!}.$$

Since the formal Borel transform satisfies $\mathcal{B}[\tilde{A} \cdot \tilde{B}](s) = \mathcal{B}[\tilde{A}] * \mathcal{B}[\tilde{B}](s)$, it respects the ring structure of $\mathbb{C}[[t^{-1}]]$. However it is not a ring homomorphism since $\mathcal{B}[1](s)$ is not defined. Moreover, even if we restrict our view on $\mathbb{C}[[s]]$ it is not obvious how to define a convolutive unit there. So we consider $\mathbb{C}[[s]]$ as a ring without identity.

Operations on $\mathbb{C}[[t^{-1}]]$ can be pulled back into $\mathbb{C}[[s]]$. We get the following lemma.

Lemma 2.4. *Let $\tilde{A} \in \mathbb{C}[[t^{-1}]]$. Assuming that both sides are well defined and defining $T_c[\tilde{A}](t) = \tilde{A}(t+c)$, we get*

- $\mathcal{B}[d_t \tilde{A}](s) = -s \mathcal{B}[\tilde{A}](s),$
- $\mathcal{B}[d_t^{-1} \tilde{A}](s) = -\frac{1}{s} \mathcal{B}[\tilde{A}](s),$
- $\mathcal{B}[T_c \tilde{A}](s) = e^{-cs} \mathcal{B}[\tilde{A}](s)$ for all $c \in \mathbb{C},$
- $\mathcal{B}[t \tilde{A}](s) = d_s \mathcal{B}[\tilde{A}](s).$

Remark. Using the above lemma one could write

$$\mathcal{B}[1](s) = \mathcal{B}[t \cdot \frac{1}{t}](s) = d_s \mathcal{B}[\frac{1}{t}](s) = d_s 1 = 0.$$

This hints that a convolutive unit cannot be defined using this definition of Borel transform.

The space $\mathbb{C}[[s]]$ happens to be too big for our needs, so we consider a smaller one, namely $\mathbb{C}\{s\}$, the space of convergent series around the origin. We have that

$$\hat{A}(s) = \sum_{n \geq 0} a_n \frac{s^n}{n!} \in \mathbb{C}\{s\}$$

if and only if there exist $M, \alpha > 0$ such that for all $n \in \mathbb{N}$, $a_n \leq M \alpha^n n!$. The fact that $\mathcal{B}^{-1}[\hat{A}](t) = \sum_{n \geq 0} a_n t^{-n-1}$ motivates the following definition.

Definition 2.5. Let $\mathbb{C}[[t^{-1}]]_1$ denote the space of all formal power series $\sum_{n \geq 0} a_n t^{-n}$ for which there exist $M, \alpha > 0$ such that $a_n \leq M \alpha^n n!$ for all $n \in \mathbb{N}$. This space will be called the space of *Gevrey-1 formal power series*.

2.5 Resurgence

We will introduce the idea of resurgent functions as a way to define $\mathcal{B}[1](s)$. This is not to imply that this was the historical reason, however it helps by placing the theory into context.

In order to define $\mathcal{B}[1](s)$ we need to extend the classical definition of the Laplace transform and to this end we define first the space of singularities.

2.5.1 The space of singularities

In order to define the space of singularities we need some technical definitions. First we need the definition of the Riemann surface of the logarithm and then the definition of a spiraling neighbourhood of the origin.

The Riemann surface of the logarithm

By the Riemann surface of the logarithm, let it be denoted by $\tilde{\mathbb{C}}$, we mean the universal cover of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with base point at 1. In other words, we consider the set \mathbb{P} of all paths $\gamma : [0, 1] \rightarrow \mathbb{C}^*$ with $\gamma(0) = 1$ with the equivalence relation \sim of homotopy with fixed endpoints, namely

$$\begin{aligned} \gamma_0 \sim \gamma_1 &\iff \exists H : [0, 1] \times [0, 1] \rightarrow \mathbb{C}^* \text{ continuous, with } H(0, \cdot) = \gamma_0, H(1, \cdot) = \gamma_1, \\ &H(\sigma, 0) = \gamma_0(0), H(\sigma, 1) = \gamma_0(1) \forall \sigma \in [0, 1]. \end{aligned}$$

Since the endpoint $\gamma(1)$ depends only on the equivalence class and not on the chosen representative, we can define the projection

$$\pi : \tilde{\mathbb{C}} \rightarrow \mathbb{C}^*, \gamma \mapsto \gamma(1).$$

To define a Riemann surface structure on $\tilde{\mathbb{C}}$ we need first to define a Hausdorff topology. This is done by taking a basis $\{\tilde{\mathbb{D}}_r(\gamma) | \gamma' \in \tilde{\mathbb{C}}, |\pi(\gamma) - \pi(\gamma')| < r\}$ with $\tilde{\mathbb{D}}_r(\gamma)$ the set of the equivalence classes on $\tilde{\mathbb{C}}$ classes of all paths γ' obtained as concatenation of a representative of γ and a line segment starting from $\pi(\gamma)$ contained in $\mathbb{D}_r(\pi(\gamma))$, the open disk of radius r centered at $\pi(\gamma)$. Then each basis element, π , induces a homeomorphism $\pi_{\gamma, r} : \tilde{\mathbb{D}}_r(\gamma) \rightarrow \mathbb{D}_r(\pi(\gamma))$ and that for two basis element with nonempty intersection the map $\pi_{\gamma', r'} \circ \pi_{\gamma, r}^{-1}$ is the identity on the intersection. Then we get an atlas $\{\pi_{\gamma, r}\}$ which defines an 1-dimensional complex

manifold structure on $\tilde{\mathbb{C}}$.

Note that the fact that the base point is at 1 plays no special role in the construction of the surface and can be moved to any other point of \mathbb{C}^* .

An alternative way to construct this surface is through the exponential function, by defining $\tilde{\mathbb{C}} := \exp(\mathbb{C})$ such that $\exp^{-1} : \tilde{\mathbb{C}} \rightarrow \mathbb{C}$ is a bijection. This method has the advantage that we do not need to choose an arbitrary point in $\tilde{\mathbb{C}}$ to act as a base. However the construction through homotopies of path gives more insight into notions that will follow.

Spiraling neighbourhoods of the origin

A spiraling neighbourhood of the origin is in essence what we get if on the Riemann surface of the logarithm we restrict the distance we can move away from the origin.

Let $h : \mathbb{R} \rightarrow (0, \infty)$ be continuous, then we define \mathfrak{P} as the set of all paths of the form $\gamma : [0, 1] \rightarrow \mathbb{C}^*$ with $\gamma(s) = r(\theta s)e^{i\theta s}$, for some $\theta \in \mathbb{R}$ and r continuous, such that $0 < r(\sigma) < h(\sigma)$, $\sigma \in \mathbb{R}$ and $r(0) = h(0)/2$. As above, we consider the set of all homotopy classes of \mathfrak{P} and we call it a spiraling neighbourhood, \mathcal{V}_h . As above, $\mathcal{V}(h)$ can be given a local 1-dimensional complex manifold structure.

Similarly to $\tilde{\mathbb{C}}$, there is an alternative definition of $\mathcal{V}(h)$ through the exponential function. For this we fix $H : \mathbb{R} \rightarrow \mathbb{R}$ and we define

$$\mathbb{C}_H := \{z \in \mathbb{C} : \operatorname{Re}(z) < H(\operatorname{Im}(z))\}$$

and

$$\mathcal{V}(h) := \exp(\mathbb{C}_H)$$

with $h = \exp \circ H$. As noted above, we will use the definition by homotopy classes of paths since it gives more insight in the present analysis.

The space of singularities

We denote by ANA the space of functions analytic in a spiraling neighbourhood of the origin. Formally we have the following definition.

Definition 2.6. We consider the space of all pairs (\check{f}, h) , with $h : \mathbb{R} \rightarrow (0, \infty)$ continuous and $\check{f} : \mathcal{V}(h) \rightarrow \mathbb{C}$ analytic, equipped with the equivalence relation

$$(\check{f}_1, h_1) \sim (\check{f}_2, h_2) \iff \check{f}_1 \equiv \check{f}_2 \text{ on } \mathcal{V}(\min\{h_1, h_2\}).$$

Then we define the space ANA as the quotient set.

We see by the above definition that functions which are analytic in a neighbourhood of the origin are contained in the space ANA. To get the space of singularities we need to mod out all regular functions from ANA.

Definition 2.7. We define the space $SING = ANA/\mathbb{C}\{s\}$ and we denote the quotient map by

$$\text{sing}_0 : ANA \rightarrow SING, \check{f} \mapsto \text{sing}_0(\check{f}) = \mathring{f}.$$

Any representative \check{f} of \mathring{f} is called a *major* of \mathring{f} .

Example 2.8. We have $\text{sing}_0(\frac{1}{e^s-1}) = \text{sing}_0(\frac{1}{s})$ and $\text{sing}_0(\frac{s}{e^s-1}) = 0$.

Definition 2.9. The linear map defined by¹

$$\text{var} : SING \rightarrow ANA, \mathring{f} \mapsto \check{f}(s) - \check{f}(se^{-2\pi i})$$

is called variation and $\hat{f} = \text{var}(\mathring{f})$ is called the *minor* of \mathring{f} .

Example 2.10. Let $\phi \in \mathbb{C}\{s\}$. Then $\text{var}(\text{sing}_0(\phi(s) \log(s))) = 2\pi i \phi(s)$.

The kernel of var is the space of all power series in s^{-1} convergent around the origin.

The algebra of singularities

The space SING can be turned into a convolutive algebra with a properly defined convolution.

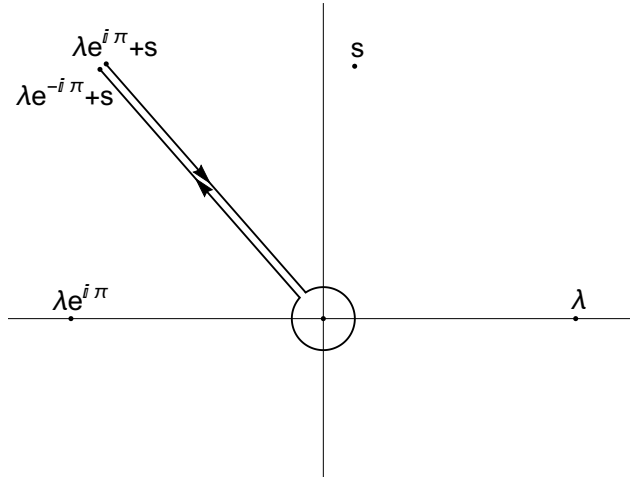
Definition 2.11. Let $\mathring{f}_1, \mathring{f}_2 \in SING$, with $(\check{f}_1, h_1), (\check{f}_2, h_2) \in ANA$. Then choose λ such that $\lambda, \lambda e^{\pi i} \in \mathcal{V}(\min\{h_1, h_2\})$ and let

$$H_\lambda = \{s \in \mathcal{V}(\min\{h_1, h_2\}) \mid \arg \lambda < \arg s < \arg \lambda + \pi\}.$$

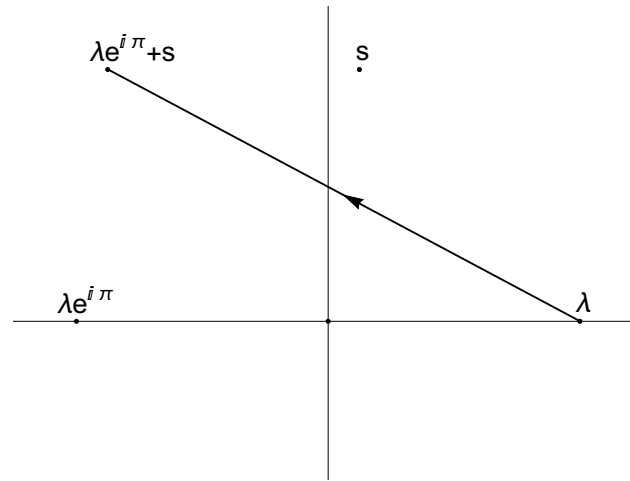
Then for $s \in H_\lambda$ with $|s|$ small enough we define

$$\mathring{f}_1 * \mathring{f}_2(s) = \text{sing}_0 \left(\int_{\Gamma_{\lambda,1}} \check{f}_1(\sigma) \check{f}_2(s - \sigma) d\sigma \right),$$

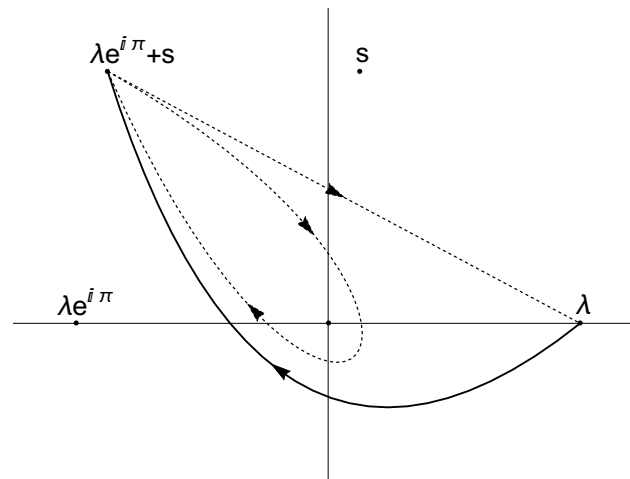
¹ There is an obvious abusion of notation here. Since $e^{-2\pi i} = 1$, it is expected that $se^{-2\pi i} = s$. However instead of this obvious choice one should think of $se^{-2\pi i}$ as the concatenation of 2 paths. One belonging to the homotopy class of s and the other going from $\pi(s)$ to $\pi(s)$ clockwise around the origin. Alternatively if one uses the definition of $\mathcal{V}(h)$ by the exponential function, then there exists $\sigma \in \mathbb{C}$ such that $s = \exp(\sigma)$. In this case $se^{-2\pi i}$ should be thought as $\exp(\sigma - 2\pi i)$.



(a) $\Gamma_{\lambda,1}$.



(b) $\Gamma_{\lambda,2}$.



(c) The difference of $\Gamma_{\lambda,1}$ and $\Gamma_{\lambda,2}$.

Figure 2.4: From top to bottom: $\Gamma_{\lambda,1}$, $\Gamma_{\lambda,2}$ and their difference.

with $\Gamma_{\lambda,1}$ as shown in Figure 2.4, and we call that the *convolution* of \mathring{f}_1 and \mathring{f}_2 .

Similarly, we can define

$$\mathring{f}_1 \underset{2}{*} \mathring{f}_2(s) = \text{sing}_0 \left(\int_{\Gamma_{\lambda,2}} \check{f}_1(\sigma) \check{f}_2(s - \sigma) d\sigma \right),$$

with $\Gamma_{\lambda,2}$ as shown in Figure 2.4. We will see that these two definitions coincide in SING.

Notice that in Figure 2.4 where $\Gamma_{\lambda,1}$ is shown, the points $\lambda e^{\pi i} + s$ and $\lambda e^{-\pi i} + s$ are drawn as two distinct points for clarity even though they have the same projection. Similarly the line segments to these points are drawn as distinct.

It is not hard to see that the two definitions of convolution coincide. Let's consider their difference $\mathring{f}_1 \underset{2}{*} \mathring{f}_2(s) - \mathring{f}_1 \underset{1}{*} \mathring{f}_2(s)$. We see that this difference is an integral over a line that has both 0 and s at the same side. This means that s can be pushed to 0 without problems which implies that the difference is analytic around the origin, hence the two definitions coincide in SING. Similarly the definition of convolution does not depend on λ .

The convolution defined on SING is linear and symmetric. Moreover, we can easily guess a unit for this algebra by the Riemann integral, i.e. $\delta(s) = \text{sing}_0(\frac{1}{2\pi i s})$. Using the first definition of convolution we get $\delta * \mathring{f}(s) = \mathring{f}(s)$.

Multiplication of resurgent functions

Naturally we would like to extend the ring of resurgent functions to allow multiplication. However we will see that the product of two resurgent functions cannot be defined uniquely.

Let \mathring{f} and \mathring{g} be singularities and let ϕ and ψ be functions analytic around the origin. We could define

$$\mathring{f} \cdot \mathring{g} = \text{sing}_0(\check{f} \cdot \check{g}).$$

The problem that arises is that $\check{f} + \phi$ is also a major of \mathring{f} so equally we have

$$\begin{aligned} \mathring{f} \cdot \mathring{g} &= \text{sing}_0((\check{f} + \phi) \cdot (\check{g} + \psi)) \\ &= \text{sing}_0(\check{f} \cdot \check{g} + \psi \cdot \check{f} + \phi \cdot \check{g} + \phi \cdot \psi) \\ &= \text{sing}_0(\check{f} \cdot \check{g} + \psi \cdot \check{f} + \phi \cdot \check{g}). \end{aligned}$$

This shows that the product defined this way depends on the majors.

We can define the product of a singularity and an analytic function in a unique way,

$$\phi \cdot \mathring{f} = \text{sing}_0(\phi \cdot \check{f}).$$

However it holds that

$$s(\mathring{f} * \mathring{g}) = (s\mathring{f}) * \mathring{g} + \mathring{f} * (s\mathring{g}),$$

which means that the multiplication by s is a derivation for this algebra. This means that the multiplication by an analytic function should be thought as the application of a differential operator of infinite order.

Correspondingly multiplication by $\frac{1}{s}$ acts as an integration. For a singularity \mathring{f} we define

$$\frac{1}{s}\mathring{f} = \text{sing}_0\left(\frac{1}{s}(\check{f} + \phi)\right) = \text{sing}_0\left(\frac{1}{s}\check{f}\right) + \phi(0)\delta.$$

Since ϕ is arbitrary, $\phi(0)\delta$ has the role of the integration constant.

We can define uniquely the multiplication of $\frac{1}{s}$ with simple singularities by defining

$$P\left(\frac{1}{s}\right) \cdot \mathring{\phi} = \text{sing}_0\left(\frac{1}{2\pi i} P\left(\frac{1}{s}\right) \cdot \phi \cdot \log\right)$$

and

$$P\left(\frac{1}{s}\right) \cdot \text{sing}_0\left(Q\left(\frac{1}{s}\right)\right) = \text{sing}_0\left(P\left(\frac{1}{s}\right) \cdot Q\left(\frac{1}{s}\right)\right)$$

with P and Q a polynomials.

2.5.2 The generalized Borel and Laplace transforms

Let $\mathring{f} \in \text{SING}$ and take a major (or a representative of the class) $\check{f} \in \text{ANA}$. We will assume that for some $\theta \in [0, 2\pi]$ \check{f} can be continued analytically along a neighbourhood of the half-line $e^{i\theta}\mathbb{R}^+$ on two neighbouring sheets and its variation is of exponential type along this line. Then we define

$$\mathcal{L}^\theta \mathring{f}(t) = \int_{\Gamma_\theta} e^{-st} \check{f}(s) ds$$

where the path Γ_θ coming from infinity on the half-line $e^{\theta i}\mathbb{R}^+$, circulating around the origin and then going to infinity along $e^{\theta i}\mathbb{R}$. In Figure 2.5 this path is shown with the two half lines separated for clarity.

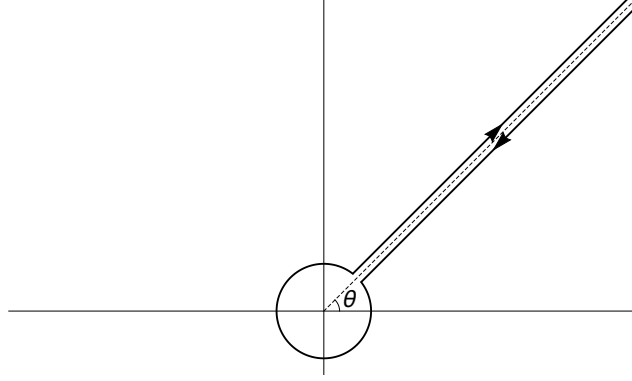


Figure 2.5: The path Γ_θ used in the definition of the generalized Laplace transform.

Clearly the generalized Laplace transform does not depend on the major that is chosen, so this is actually a definition for the Laplace transform of elements of SING. We just need to check that this definition is compatible with the classical one.

Let $\hat{\phi}$ be a function analytic around the origin and of exponential growth along the half-line $e^{\theta i}\mathbb{R}^+$. We define

$$b[\hat{\phi}](s) = \hat{\phi}(s) \frac{\log(s)}{2\pi i} \text{ and } {}^b\hat{\phi} = \text{sing}_0(b[\hat{\phi}]).$$

This can be considered as an embedding of $\mathbb{C}\{s\}$ into SING, because it satisfies ${}^b(\hat{\phi} * \hat{\psi}) = {}^b\hat{\phi} * {}^b\hat{\psi}$. Then the classical Laplace transform, $\mathcal{L}^\theta[\hat{\phi}](t) = \int_0^{\infty e^{i\theta}} e^{-st} \hat{\phi}(s) ds$, and the generalized Laplace transform, $\mathcal{L}^\theta[{}^b\hat{\phi}]$, coincide. This implies the map b is the canonical embedding of $\mathbb{C}\{s\}$ into SING and allows us to abuse the notation of the 2 different Laplace transforms. We define

$$\text{ANA}^{\text{reg}} = b(\mathbb{C}\{s\}) \quad \text{and} \quad \text{SING}^{\text{reg}} = {}^b\mathbb{C}\{s\} = \text{ANA}^{\text{reg}}/\mathbb{C}\{s\}.$$

The formal Borel transform is defined analogously to the classical case as

$$\mathcal{B}[t^{-n-1}](s) = \frac{{}^b s^n}{n!} = \text{sing}_0\left(\frac{s^n \log(s)}{n! \cdot 2\pi i}\right).$$

Now for any $\theta \in \mathbb{R}$ we have $\mathcal{L}^\theta[\frac{1}{2\pi i s}](t) = \int_{\Gamma_\theta} e^{-st} \frac{1}{2\pi i s} ds = 1$, so we can define $\check{\delta}(s) = \frac{1}{2\pi i s}$ and $\mathcal{B}[1](s) = \delta(s) = \text{sing}_0(\check{\delta}(s))$. Similarly we define $\check{\delta}^{(n)}(s) = \frac{(-1)^n n!}{2\pi i s^{n+1}}$

and $\mathcal{B}[t^n](s) = \delta^{(n)}(s) = \text{sing}_0(\check{\delta}^{(n)}(s))$ for any $n \in \mathbb{N}$.

2.5.3 The Borel transform of Gevrey-1 formal series.

With the above generalization of the Borel transform we can map any element of $\mathbb{C}[[t^{-1}]]_1$ to ANA.

For $\tilde{\Phi} \in \mathbb{C}[[t^{-1}]]_1$ with $\tilde{\Phi}(t) = c + \tilde{\phi}(t)$, $\tilde{\phi} \in t^{-1}\mathbb{C}[[t^{-1}]]_1$ we define

$$\mathcal{B}[\tilde{\Phi}](s) = c\check{\delta}(s) + \hat{\phi}(s)\frac{\log(s)}{2\pi i} = \check{\Phi}(s),$$

with $\hat{\phi}(s) \in \mathbb{C}\{s\}$ given by the classical Borel transform. By taking the quotient we can map $\tilde{\Phi}$ to SING, so we define

$$\mathcal{B}[\tilde{\Phi}] = c\delta + {}^b\hat{\phi} = \check{\Phi}.$$

Due to the properties of the Borel transform we see that it is a ring isomorphism from $\mathbb{C}[[t^{-1}]]_1$ to $\mathcal{B}[\mathbb{C}[[t^{-1}]]_1]$.

We define

$$\mathbb{C}[t][[t^{-1}]]_1 = \left\{ P(t) + \tilde{\Phi}(t) \mid \tilde{\Phi} \in \mathbb{C}[[t^{-1}]]_1, \exists n \in \mathbb{N}, P(t) = \sum_{k=1}^n p_k t^k \right\}$$

and

$$\text{ANA}^{\text{sim}} = \left\{ P[\check{\delta}] + \hat{\phi} \mid \exists n \in \mathbb{N}, P[\delta] = \sum_{k=1}^n p_k \check{\delta}^{(k)}, \exists \tilde{\phi} \in \mathbb{C}[[t^{-1}]]_1, \mathcal{B}[\tilde{\phi}] = \hat{\phi} \right\}.$$

The space of *simple singularities* is defined by the quotient

$$\text{SING}^{\text{sim}} = \text{ANA}^{\text{sim}} / \mathbb{C}\{s\}.$$

Then the following are true:

$$\begin{aligned} b : \mathbb{C}\{s\} &\rightarrow \text{ANA}^{\text{reg}} \subset \text{ANA}^{\text{sim}}, \\ \text{sing}_0 : \text{ANA}^{\text{sim}} &\rightarrow \text{SING}^{\text{sim}}, \\ \text{var} : \text{SING}^{\text{sim}} &\rightarrow \mathbb{C}\{s\} \end{aligned}$$

and the map

$$\text{var} \circ \text{sing}_0 \circ b : \mathbb{C}\{s\} \rightarrow \mathbb{C}\{s\}$$

is the identity map.

This happens because an element of ANA^{sim} can have 3 “components”: a convergent power series of s , a polynomial of s^{-1} and a logarithmic branching. The map sing_0 , which is the quotient map, kills the convergent power series. Then the variation kills the polynomial and gives the difference of 2 consecutive branches of the logarithm, which is a regular function.

Remark. The statement of Lemma 2.4 holds in the general case. This can be seen by considering the non-formal inverse Laplace transform. So in particular, it holds when we consider the space $\mathbb{C}[t][[t^{-1}]]_1$. For example we have

$$\begin{aligned}\mathcal{B}[t^{-n}] &= \mathcal{B}[t \cdot t^{-n-1}] = \text{sing}_0 \left[d_s \frac{s^n \log(s)}{n! \cdot 2\pi i} \right] = \text{sing}_0 \left[\frac{s^{n-1}}{2\pi i n!} + \frac{s^{n-1}}{(n-1)!} \frac{\log(s)}{2\pi i} \right] \\ &= \text{sing}_0 \left[\frac{s^{n-1}}{(n-1)!} \frac{\log(s)}{2\pi i} \right].\end{aligned}$$

Also

$$\mathcal{B}[1] = \mathcal{B}[t \cdot t^{-1}] = \text{sing}_0 \left[d_s \frac{\log(s)}{2\pi i} \right] = \text{sing}_0 \left[\frac{1}{2\pi i s} \right] = \delta.$$

Remark. When we consider an \check{f} in SING^{sim} we need not to define the value of some \check{f} on the base point of its spiraling neighbourhood, $\mathcal{V}(h_{\check{f}})$. This is because the variation of \check{f} , i.e. the difference of two consecutive branches of \check{f} , is always the same regular function no matter where we are on $\mathcal{V}(h_{\check{f}})$.

2.5.4 The Riemann surfaces \mathcal{R}_1 and \mathcal{R}_0

In this section we define two Riemann surfaces that are instrumental to the analysis. The first one, \mathcal{R}_1 , is the universal cover of $\mathbb{C} \setminus 2\pi i\mathbb{Z}$. Formally we have the following definition.

Definition 2.12. Let \mathcal{R}_1 be the set of all homotopy classes of continuously differentiable paths $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus 2\pi i\mathbb{Z}$ with $\gamma(0) = 1$ and $|\dot{\gamma}| = |\gamma|$. Let $\pi : \mathcal{R}_1 \rightarrow \mathbb{C} \setminus 2\pi i\mathbb{Z}$, $\gamma \mapsto \gamma(1)$ be the projection map. We consider \mathcal{R}_1 as a Riemann surface by pulling back through π the complex structure of $\mathbb{C} \setminus 2\pi i\mathbb{Z}$.

As in the Riemann surface of the logarithm, the base point 1 is not special and can be moved to any other point of $\mathbb{C} \setminus 2\pi i\mathbb{Z}$. We define the principal sheet of \mathcal{R}_1 , denoted by \mathcal{R}_1^{p} , as the set of all homotopy classes of paths $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus (\mathbb{R}_0^- \cup \pm 2\pi i[1, \infty))$, $\gamma(0) = 1$.

The second surface, \mathcal{R}_0 , is essentially \mathcal{R}_1 plus the origin. This means that we consider all paths that have a base point at 0 instead of 1. Formally we have the following definition.

Definition 2.13. Let \mathcal{R}_0 be the set of all homotopy classes of continuous and piecewise continuously differentiable paths $\gamma : [0, 1] \rightarrow \mathbb{C}$ with $\gamma(0) = 0$, $\gamma((0, 1]) \subset \mathbb{C} \setminus 2\pi i\mathbb{Z}$ and $|\dot{\gamma}| = |\gamma|$ plus the path $\mathbf{0} : [0, 1] \rightarrow 0$. Let $\pi : \mathcal{R}_0 \rightarrow \mathbb{C} \setminus 2\pi i\mathbb{Z}^*$, $\gamma \mapsto \gamma(1)$ be the projection map. We consider \mathcal{R}_0 as a Riemann surface by pulling back through π the complex structure of $\mathbb{C} \setminus 2\pi i\mathbb{Z}^*$.

Notice that the preimage of the origin is just one point of \mathcal{R}_0 , namely $\pi^{-1}(0) = \{\mathbf{0}\}$. Because of this the base point 0 is special for \mathcal{R}_0 and cannot be moved. Informally \mathcal{R}_0 can be viewed as \mathcal{R}_1 with the origin attached. Similarly to \mathcal{R}_1 we define the principal sheet of \mathcal{R}_0 , denoted by \mathcal{R}_0^p , to be the set of all homotopy classes of paths $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus (\pm 2\pi i[1, \infty))$, $\gamma(0) = 0$.

We can now consider the space $\mathbb{C}^\omega(\mathcal{R}_0)$. This is the space of functions analytic on \mathcal{R}_0 , which implies that they are also analytic around the origin. Similarly the space $\mathbb{C}^\omega(\mathcal{R}_1)$ is defined to be the space of functions analytic on \mathcal{R}_1 .

2.5.5 Resurgent functions

Resurgent functions should be thought of as singularities that can be extended analytically along paths that avoid some points on \mathbb{C} , where other singularities appear. It is interesting to note that for any $\mathring{f} \in \text{SING}$ only the variation \hat{f} plays a role when the singularity needs to be continued beyond its initial domain of definition. This happens since any regular part around 0 that may have singularities elsewhere is killed by the quotient and the part that depends on s^{-1} can only have a singularity at 0. So only the variation can create singularities away from the origin.

In general the set of singular points is not defined a priori. However for the present analysis, we do not need the theory in its full generality so we will predefine this set of singularities. Usual choices for applications are $2\pi i\mathbb{Z}$, \mathbb{N} and \mathbb{Z} . Here the first option will be used. This means that from now on, every time that resurgent functions are considered, it is implicitly assumed that they are functions with singularities at $2\pi i\mathbb{Z}$. This leads us to the following definition.

Definition 2.14. The space of resurgent functions RES is defined as the quotient

$$\widehat{\text{RES}} = \mathbb{C}^\omega(\mathcal{R}_1) / \mathbb{C}\{s\}.$$

We have the inclusion map $\text{sing}_0 : \widehat{\text{RES}} \rightarrow \text{SING}$ and we define $\text{res} : \text{SING} \rightarrow \widehat{\text{RES}}$ as the inverse of sing_0 on its image. With this we can pull back convolution and variation from SING in $\widehat{\text{RES}}$. The variation is trivially generalized in $\widehat{\text{RES}}$, however there is an obvious problem with convolution. In SING convolution is well defined only close to 0. So it needs to be extended in a consistent way. This is done by constructing symmetric paths, see Section 2.5.7. Using symmetric paths we get that if Ω is a discrete subset of \mathbb{C} that is invariant under addition then the space of resurgent functions that have singularities on Ω are invariant under convolution, see [Sau13b]. Obviously this is the case with $2\pi i\mathbb{Z}$.

Definition 2.15. We define the space of *regular resurgent functions* as

$$\widehat{\text{RES}}^{\text{reg}} = \text{res} \circ \text{sing}_0 \circ b(\mathbb{C}^\omega(\mathcal{R}_0)) \subset \widehat{\text{RES}}.$$

In simple words this definition means that we take an element of $\mathbb{C}^\omega(\mathcal{R}_0)$ and multiply it by $\frac{\log(s)}{2\pi i}$ which makes it an element of $\mathbb{C}^\omega(\mathcal{R}_1)$. Then by applying sing_0 we get the quotient with $\mathbb{C}\{s\}$. This is a singularity that can be extended analytically to \mathcal{R}_1 so by definition is in $\text{sing}_0(\widehat{\text{RES}})$. Then the map res is well defined, so finally we get an element of $\widehat{\text{RES}}$. Evidently the map $\text{var} : \widehat{\text{RES}}^{\text{reg}} \rightarrow \mathbb{C}^\omega(\mathcal{R}_0)$ is the inverse of $\text{res} \circ \text{sing}_0 \circ b$ and we see that $\widehat{\text{RES}}^{\text{reg}}$ is isomorphic to $\mathbb{C}^\omega(\mathcal{R}_0)$. This motivates the following definition.

Definition 2.16. We define the space of *regular resurgent power series* as

$$\widetilde{\text{RES}}^{\text{reg}} = \mathcal{B}^{-1} \left[\widehat{\text{RES}}^{\text{reg}} \right] \subset \frac{1}{t} \mathbb{C}[[\frac{1}{t}]]_1.$$

To extend the algebraic structure of SING to $\widehat{\text{RES}}^{\text{reg}}$ we use its isomorphism with $\mathbb{C}^\omega(\mathcal{R}_0)$. On $\mathbb{C}^\omega(\mathcal{R}_0)$ we can use the classical definition of convolution close to the origin and we can extend it using symmetric paths. See next section. This implies that we can pull back the operation of convolution on $\widehat{\text{RES}}^{\text{reg}}$ and that this space is also stable under it. So both spaces are algebras with their respective convolutions but without unit. Then the space $\widetilde{\text{RES}}^{\text{reg}}$ is stable under multiplication and of course this makes it also an algebra without unit.

We can extend these spaces to turn them into algebras with units.

Definition 2.17. We define the space of *simple resurgent functions* as

$$\widehat{\text{RES}}^{\text{sim}} = \mathbb{C} \left[\frac{1}{s} \right] \oplus \widehat{\text{RES}}^{\text{reg}},$$

with $\mathbb{C} \left[\frac{1}{s} \right]$ being the space of polynomials in $\frac{1}{s}$.

We can define convolution in $\widehat{\text{RES}}^{\text{sim}}$ in a similar way by noting that elements of $\mathbb{C} \left[\frac{1}{s} \right]$ are analytic in \mathbb{C}^* so convolution between them and elements of $\widehat{\text{RES}}^{\text{reg}}$ can be treated as a convolution in SING.

Example 2.18. Let

$$\mathring{f}(s) = \alpha \delta^{(1)}(s) + \beta \delta(s) \text{ and } \mathring{g}(s) = {}^b\phi(s), \quad \phi \in \mathbb{C}\{s\}.$$

Then

$$\begin{aligned} \check{f} * \check{g}(s) &= \left(\alpha \check{\delta}^{(1)}(s) + \beta \check{\delta}(s) \right) * \left(\phi(s) \frac{\log(s)}{2\pi i} \right) \\ &= \alpha \check{\delta}^{(1)} * \left(\phi(s) \frac{\log(s)}{2\pi i} \right) + \beta \check{\delta} * \left(\phi(s) \frac{\log(s)}{2\pi i} \right) \\ &= a \, d_s \left(\phi(s) \frac{\log(s)}{2\pi i} \right) + b \phi(s) \frac{\log(s)}{2\pi i} \\ &= a \frac{\phi(s)}{2\pi i s} + \alpha \phi'(s) \frac{\log(s)}{2\pi i} + \beta \phi(s) \frac{\log(s)}{2\pi i} \\ &= a \frac{\phi(0)}{2\pi i s} + \alpha \phi'(s) \frac{\log(s)}{2\pi i} + \beta \phi(s) \frac{\log(s)}{2\pi i} + a \frac{\phi(s) - \phi(0)}{2\pi i s} \\ &= a \phi(0) \check{\delta}(s) + \alpha \phi'(s) \frac{\log(s)}{2\pi i} + \beta \phi(s) \frac{\log(s)}{2\pi i} + a \frac{\phi(s) - \phi(0)}{2\pi i s}. \end{aligned}$$

Note that the function $\frac{\phi(s) - \phi(0)}{2\pi i s}$ is analytic around the origin so we have

$$\mathring{f} * \mathring{g} = a \phi(0) \delta + \alpha {}^b\phi' + \beta {}^b\phi.$$

Definition 2.19. We define the space of *simple resurgent power series* as

$$\widehat{\text{RES}}^{\text{sim}} = \mathcal{B}^{-1} \left[\widehat{\text{RES}}^{\text{sim}} \right] \subset \mathbb{C}[t][[\frac{1}{t}]]_1.$$

As a concluding remark we must say that the convolution on SING can be extended to a proper convolution in $\widehat{\text{RES}}$ by using symmetric paths with the only difference these paths cannot originate from 0.

2.5.6 Alien Calculus

We saw that sing_0 is an inclusion of $\widehat{\text{RES}}$ in SING. By definition, any element of $\widehat{\text{RES}}$ can be continued on any path originating at 1 and that avoids $2\pi i\mathbb{Z}$. This means that any element of $\widehat{\text{RES}}$ will have a singularity² at any point $\omega \in 2\pi i\mathbb{Z}$, so

²If the function is actually analytic at ω we will say that it has the 0 singularity.

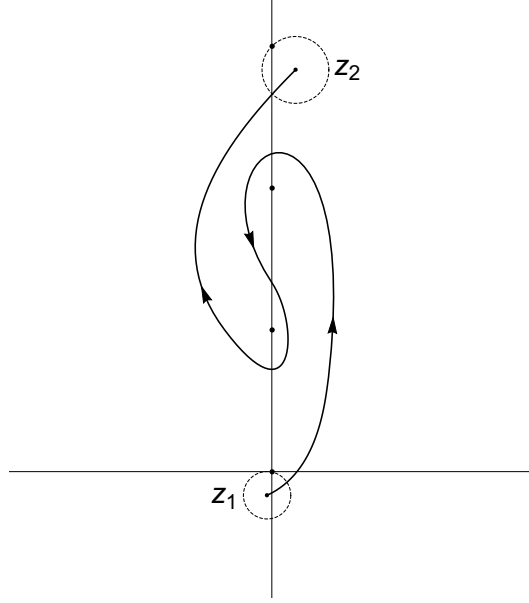


Figure 2.6: Continuation path in \mathcal{R}_1 .

one could expect that by translating the origin to ω we get an element of SING. However translation on \mathcal{R}_1 is ambiguous. For this reason we need to consider the continuation of an element of $\widehat{\text{RES}}$ along a path γ , denoted by cont_γ . Let $\hat{f} \in \widehat{\text{RES}}$ and $\gamma \in \mathcal{R}_1$. Then there exists a unique analytic continuation of \hat{f} along γ .

Let $n \in \mathbb{Z}$ and $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus 2\pi i\mathbb{Z}$ with $\gamma(0) = z_1$ not a negative real number and $\gamma(1) = z_2$, such that $|z_1| = \varepsilon_1$ and $|z_2 - \omega| = \varepsilon_2$ and $\varepsilon_1, \varepsilon_2 < \pi$, see Figure 2.6 for an example. Then the path γ can be thought of as the concatenation of two paths. The first γ_1 from 1 to z_1 with γ_1 in the principal sheet of \mathcal{R}_1 and γ_2 from 1 to z_2 . So then γ is the concatenation of the reversed γ_1 and γ_2 . It is worthy noting that in this procedure there is a hidden choice³, which is the value of the function on the principal sheet of \mathcal{R}_1 . This choice is not unique but once it is made then the continuation depends only on γ .

The analytic continuation along γ gives a function analytic in $\tilde{\mathbb{D}}_{\varepsilon_2}(\gamma_2)$ and from this we can define a function analytic on $\mathbb{D}_{\varepsilon_2}(z_2)$ since $\tilde{\mathbb{D}}_{\varepsilon_2}(\gamma_2)$ is isomorphic to $\mathbb{D}_{\varepsilon_2}(z_2)$. Now by a simple translation close to the origin to get a function analytic in $\mathbb{D}_{\varepsilon_2}(z_2 - \omega)$. Then the function can be continued in the spiraling neighbourhood of the origin $\mathcal{V}(C_{2\varepsilon_2})$, with $C_{2\varepsilon_2} : \mathbb{R} \rightarrow \{2\varepsilon_2\}$, thus we get an element of ANA. We

³There is no choice when we consider functions in ${}^b\mathbb{C}\{s\}$.

denote the procedure described above by

$$\text{cont}_\omega^\gamma : \widehat{\text{RES}} \rightarrow \text{ANA}.$$

Naturally we define

$$\text{sing}_\omega^\gamma : \widehat{\text{RES}} \rightarrow \text{SING}, \quad \text{with} \quad \text{sing}_\omega^\gamma := \text{sing}_0 \circ \text{cont}_\omega^\gamma.$$

By the definition of $\widehat{\text{RES}}$, for any $n \in \mathbb{Z}$ and for any $\mathring{f} \in \widehat{\text{RES}}$, the singularity $\text{sing}_{\omega_n}^\gamma(\mathring{f})$ can be extended into a function on \mathcal{R}_1 . So we can use the map res to get an element of $\widehat{\text{RES}}$. We define the *alien operator associated with* (ω, γ) ,

$$\mathcal{A}_\omega^\gamma : \widehat{\text{RES}} \rightarrow \widehat{\text{RES}}, \quad \text{with} \quad \mathcal{A}_\omega^\gamma = \text{res} \circ \text{sing}_0 \circ \text{cont}_\omega^\gamma.$$

There is a particularly interesting class of alien operators denoted by $\Delta_{\omega_n}^+$ with $\omega_n = 2\pi i n$. This operator denotes the continuation to ω_n by the path that circumvents all ω_m , $0 < m < n$, counterclockwise. In Figure 2.7 the path for $\Delta_{\omega_4}^+$ is shown. For the action of $\Delta_{\omega_n}^+$ on the algebra $\widehat{\text{RES}}$ we have the following lemma.

Lemma 2.20. *Let $n \in \mathbb{N}$ and $\mathring{f}, \mathring{g} \in \widehat{\text{RES}}$. Then*

$$\Delta_{\omega_n}^+[\mathring{f} * \mathring{g}] = \Delta_{\omega_n}^+[\mathring{f}] * \mathring{g} + \sum_{m=1}^{n-1} \Delta_{\omega_m}^+[\mathring{f}] * \Delta_{\omega_{m-n}}^+[\mathring{g}] + \mathring{f} * \Delta_{\omega_n}^+[\mathring{g}].$$

A similar relation holds if n is a negative integer. An indication that this is true is given in Figure 2.8. The circles in these figures indicate what is considered to be close enough to a singularity so that we can use the convolution of SING. To get the contour in Figure 2.8, we begin with s close to 0, $\text{Re } s > 0$ and the second definition of convolution and then we continue to an s close to ω_3 along a straight line. Then we see that just by breaking path of the integral $\int_\Gamma \check{f}(\sigma) \check{g}(s - \sigma) d\sigma$ into three, we get the relation. Notice that the function $\check{f}(\sigma)$ has singularities at ω_n and the function $\check{g}(s - \sigma)$ has singularities at $s - \omega_n$.

Remark. Since ${}^b s^n$ is singular only at the origin we have $\Delta_{\omega_n}^+ [{}^b s^n * \mathring{f}] = {}^b s^n * \Delta_{\omega_n}^+ [\mathring{f}]$.

In particular we have $\Delta_{\omega_1}^+(\mathring{f} * \mathring{g}) = \Delta_{\omega_1}^+(\mathring{f}) * \mathring{g} + \mathring{f} * \Delta_{\omega_1}^+(\mathring{g})$, thus the operator $\Delta_{\omega_1}^+$ acts on $\widehat{\text{RES}}$ as a derivation. This motivates the definition of another set of linear combinations of the operators $\Delta_{\omega_n}^+$ that act as derivations.

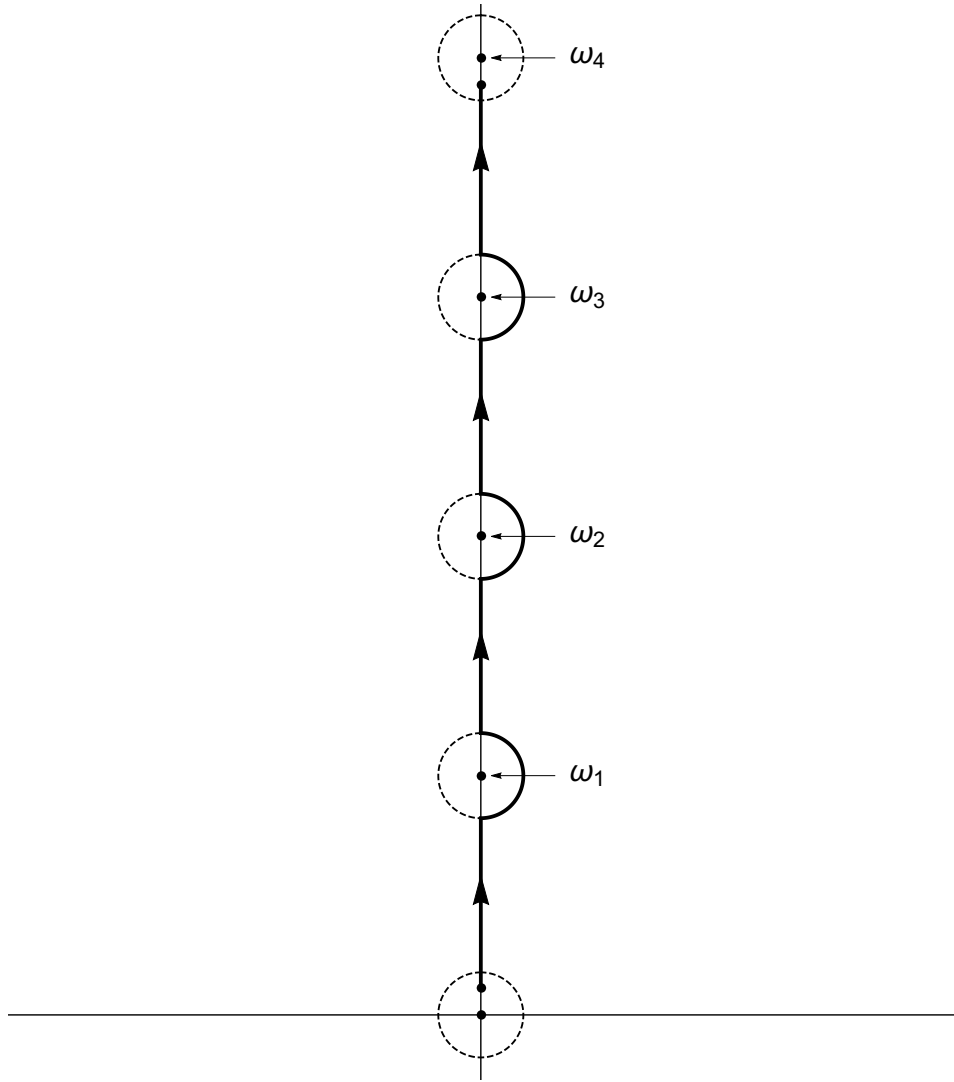


Figure 2.7: The continuation path for $\Delta_{\omega_4}^+$.

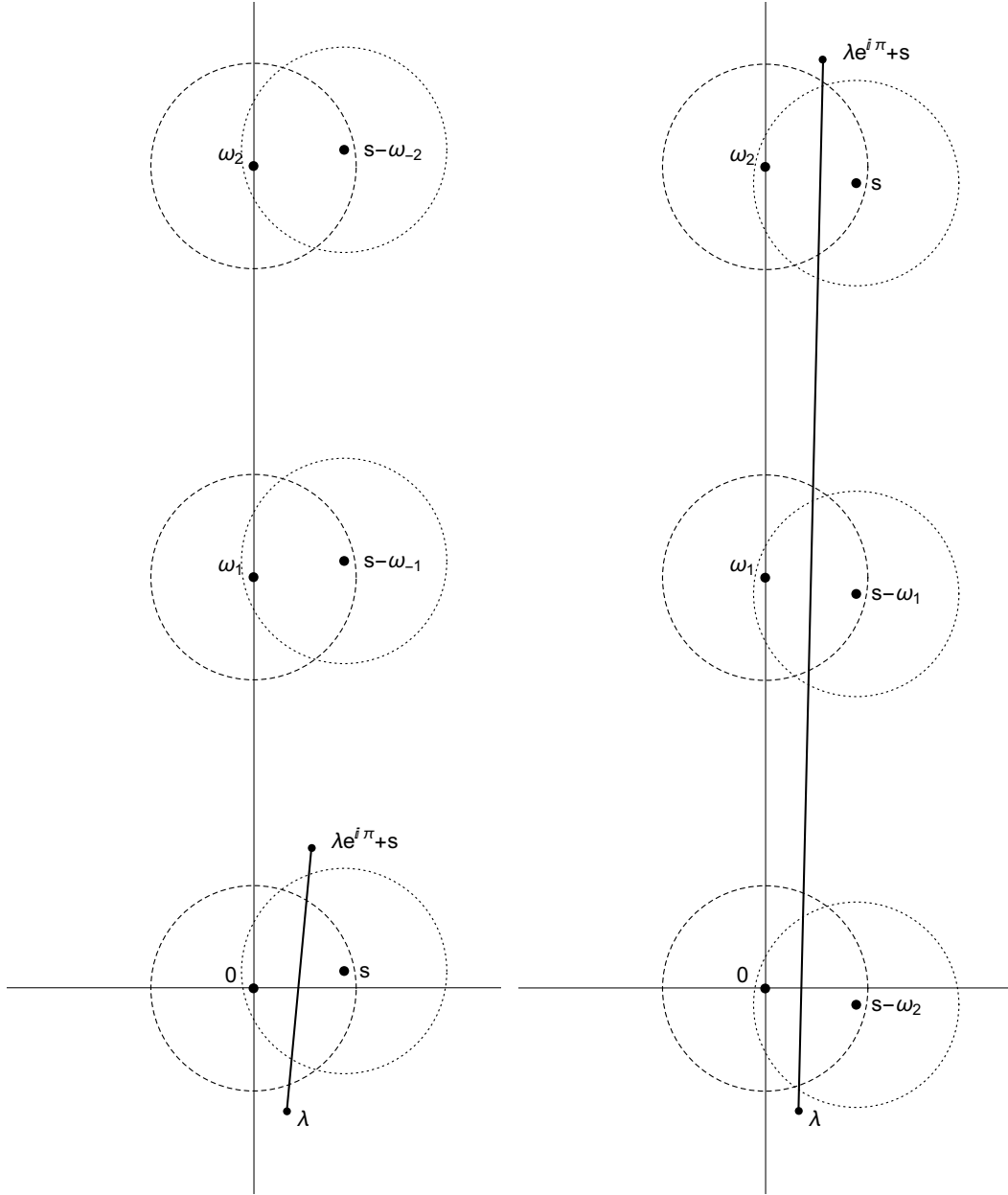


Figure 2.8: On the left the convolution path for two elements of SING it is shown. On the right s is moved on a straight line to get close to ω_2 .

Definition 2.21. For each $m \in \mathbb{N}^*$ we define

$$\Delta_{\omega_m} = \sum_{l=1}^m \frac{(-1)^l}{l} \sum_{\substack{m_1, \dots, m_l \geq 1 \\ m_1 + \dots + m_l = m}} \Delta_{\omega_{m_1}}^+ \cdots \Delta_{\omega_{m_l}}^+.$$

We define similarly Δ_{ω_m} for $m \in -\mathbb{N}^*$.

Thus we have

$$\begin{aligned} \Delta_{\omega_1} &= \Delta_{\omega_1}^+, \\ \Delta_{\omega_2} &= \Delta_{\omega_2}^+ - \frac{1}{2} \Delta_{\omega_1}^+ \Delta_{\omega_1}^+, \\ \Delta_{\omega_3} &= \Delta_{\omega_3}^+ - \frac{1}{2} (\Delta_{\omega_2}^+ \Delta_{\omega_1}^+ + \Delta_{\omega_1}^+ \Delta_{\omega_2}^+) + \frac{1}{3} \Delta_{\omega_1}^+ \Delta_{\omega_1}^+ \Delta_{\omega_1}^+, \\ &\vdots \end{aligned}$$

These new alien operators are derivations on the ring $\widehat{\text{RES}}$ and Δ_{ω} is called *alien derivative at ω* .

Knowing all the alien derivatives gives enough information to calculate any alien operator, which means that we can recover any singularity. Unfortunately the sub-rings $\widehat{\text{RES}}^{\text{reg}}$ and $\widehat{\text{RES}}^{\text{sim}}$ are not invariant under the action of the alien derivatives. This is because a regular variation at the origin cannot force any other singularity it may have. This changes if \mathring{f} satisfies some equation, see the following example.

Example 2.22. Let $F : \mathbb{C} \rightarrow \mathbb{C}$ be analytic around the origin with $F(0) = 0$. In other words, there are constants a_n such that $F(z) = \sum_{n \geq 1} a_n z^n$ in a neighbourhood of the origin. Assume that there exists $\tilde{x} \in \widehat{\text{RES}}^{\text{reg}}$ such that

$$\tilde{x}(t+1) = F(\tilde{x}(t)). \quad (2.1)$$

Let $\mathcal{B}[\tilde{x}](s) = \hat{x}(s) \in \widehat{\text{RES}}^{\text{reg}}$ and assume that \hat{x} is of exponential type along any path that does not go to infinity vertically. Then we can apply the formal Borel transform to equation (2.1) by writing

$$F(\tilde{x}(t)) = \sum_{n \geq 1} a_n \tilde{x}(t)^n \quad \text{and} \quad \mathcal{F}[\hat{x}](s) := \mathcal{B}[F(\tilde{x})](s) = \sum_{n \geq 1} a_n \hat{x}^{*n}(s).$$

Since $F'(z) = \sum_{n \geq 0} (n+1) a_{n+1} z^n$ we define

$$\mathcal{F}'[\hat{x}](s) := \mathcal{B}[F'(\tilde{x})](s) = \sum_{n \geq 1} (n+1) a_{n+1} \hat{x}^{*n}(s).$$

The Borel transform of equation (2.1) is

$$e^{-s}\hat{x}(s) = \mathcal{F}[\hat{x}](s).$$

By applying the alien derivative Δ_ω to the sum we get⁴

$$e^{-s}\Delta_\omega\hat{x}(s) = \mathcal{F}'[\hat{x}] * \Delta_\omega\hat{x}(s).$$

The formal Laplace transform of this equation is

$$\Delta_\omega\tilde{x}(t+1) = F'(\tilde{x}(t)) \Delta_\omega\tilde{x}(t),$$

with $\Delta_\omega\tilde{x}$ the formal Laplace transform of $\Delta_\omega\hat{x}$. This is a linear difference equation and obviously \tilde{x}' is a solution, so we set $\Delta_\omega\tilde{x}(t) = C(t)\tilde{x}'(t)$ and from this we get $C(t+1) = C(t)$. This implies that if we look for a solution in $\mathbb{C}[[\frac{1}{t}]]_1$ then C is a constant and we have

$$\Delta_\omega\tilde{x} = C_\omega\tilde{x}'.$$

This indicates that even though in general the singularity at 0 of an element of $\widehat{\text{RES}}^{\text{reg}}$ does not restrict any other singularities, if it is the solution of an equation, then the equation does restrict what singularities can appear. For an in depth treatment see [Sau13a, Sau08].

2.5.7 Symmetric paths

Definition 2.23. Let $\gamma : [0, 1] \rightarrow \mathbb{C}$, be continuous and piecewise continuously differentiable with $\gamma(0) = 0$. The path is parametrized with constant speed, i.e. $|\dot{\gamma}(t)| = |\gamma|$ for all t that $\dot{\gamma}(t)$ is defined. By $|\gamma|$ we denote the length of the path. If it also satisfies the relation

$$\gamma(1) - \gamma(t) = \gamma(1 - t)$$

and its homotopy class is in \mathcal{R}_0 then we say that γ is a *symmetric path* on \mathcal{R}_0 .

Note that from the above definition we get that $\gamma(\frac{1}{2}) = \frac{1}{2}\gamma(1)$. For any $\tau \in [0, 1]$ we denote

$$\gamma^\tau : t \mapsto \gamma(t\tau), \quad t \in [0, 1]$$

⁴This follows from the assumption that F is a convergent series. Because of that it cannot create any new singularities.

the path up to the point $\gamma(\tau)$. We have $\dot{\gamma}^\tau(t) = \tau \dot{\gamma}(\tau t)$, so $|\dot{\gamma}^\tau(t)| = |\gamma|\tau$ and $|\gamma^\tau| = |\gamma|\tau$. We will use also the operator notation $P_\tau : \gamma \mapsto \gamma^\tau$.

Lemma 2.24. *For each $\gamma \in \mathcal{R}_0$ there exists a symmetric path γ_{sym} such that $\gamma_{sym}((0, 1]) \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$ and $\gamma_{sym}(0) = 0$, in the same homotopy class as γ .*

For a proof of this lemma see [Sau13b]. The essence of the proof is the construction a homotopy class of paths, H , such that $H_t(0) = 0$, $H_0(t) = 0$ for all $t \in [0, 1]$, $H_t(1) = \gamma(t)$ and H_t is a symmetric path for all $t \in [0, 1]$.

There is a simple algorithm to construct such homotopy. First note that in the integral of the convolution we have the product $f(\sigma)g(s - \sigma)$, which means that f has a singularity when $\sigma = \omega_n$ but g has a singularity when $\sigma = s - \omega_n$. This indicates that the path of the integral should not cross these two sets of points.

In order to construct the homotopy we first think of the initial path γ as a rail on which a small sphere is moved. We connect the sphere to the origin with a perfectly elastic band. The set $2\pi i\mathbb{Z}^*$ is called immobile singularities and the set $z - 2\pi i\mathbb{Z}^*$ is called mobile singularities, with z the location of the sphere. Around each singularity, mobile or immobile, there is a solid disk of radius $\varepsilon > 0$ through which the elastic band cannot pass. We place the small sphere on the origin and we start moving it along the path γ . Then it is a matter of respecting the rules about where the elastic band can go and its final shape is the symmetric path. The construction is demonstrated in Figure 2.9 for a path that crosses the imaginary axes only once. The more crossings the initial path has the more complicated the symmetric path tends to be. However this does not alter the procedure.

This construction suffices when we consider the convolution in $\widehat{\text{RES}}^{\text{reg}}$. When we need to treat a bigger space we need to consider paths on \mathcal{R}_1 . From a point of view of path construction the idea is essentially the same. The only difference is what is called a symmetric path. In \mathcal{R}_1 we will call a symmetric path any path for which $\gamma(0) = \lambda$ and $\gamma(1) = \lambda e^{\pi i} + s$ where $\gamma(t) + \gamma(1 - t) = s$. So now we have the starting point λ close to the origin and the ending point $\lambda e^{\pi i} + s$ close to the point we need to evaluate the convolution at. Recall that this is the one of the two paths used to define $\mathring{f} * \mathring{g}(s)$.

In practice to construct a symmetric path in \mathcal{R}_1 , we take a path $\gamma_0 \in \mathcal{R}_0$ we begin with some z on this path close to the origin. In this case the immobile singularities are $2\pi i\mathbb{Z}$ and the mobile are the $z - 2\pi i\mathbb{Z}$. Then we imagine that there is a solid disk around them. We connect with an infinitely elastic band the points λ and $\lambda e^{\pi i} + z$. Then we move z along γ_0 and we obey the same rules as above. When z reaches s

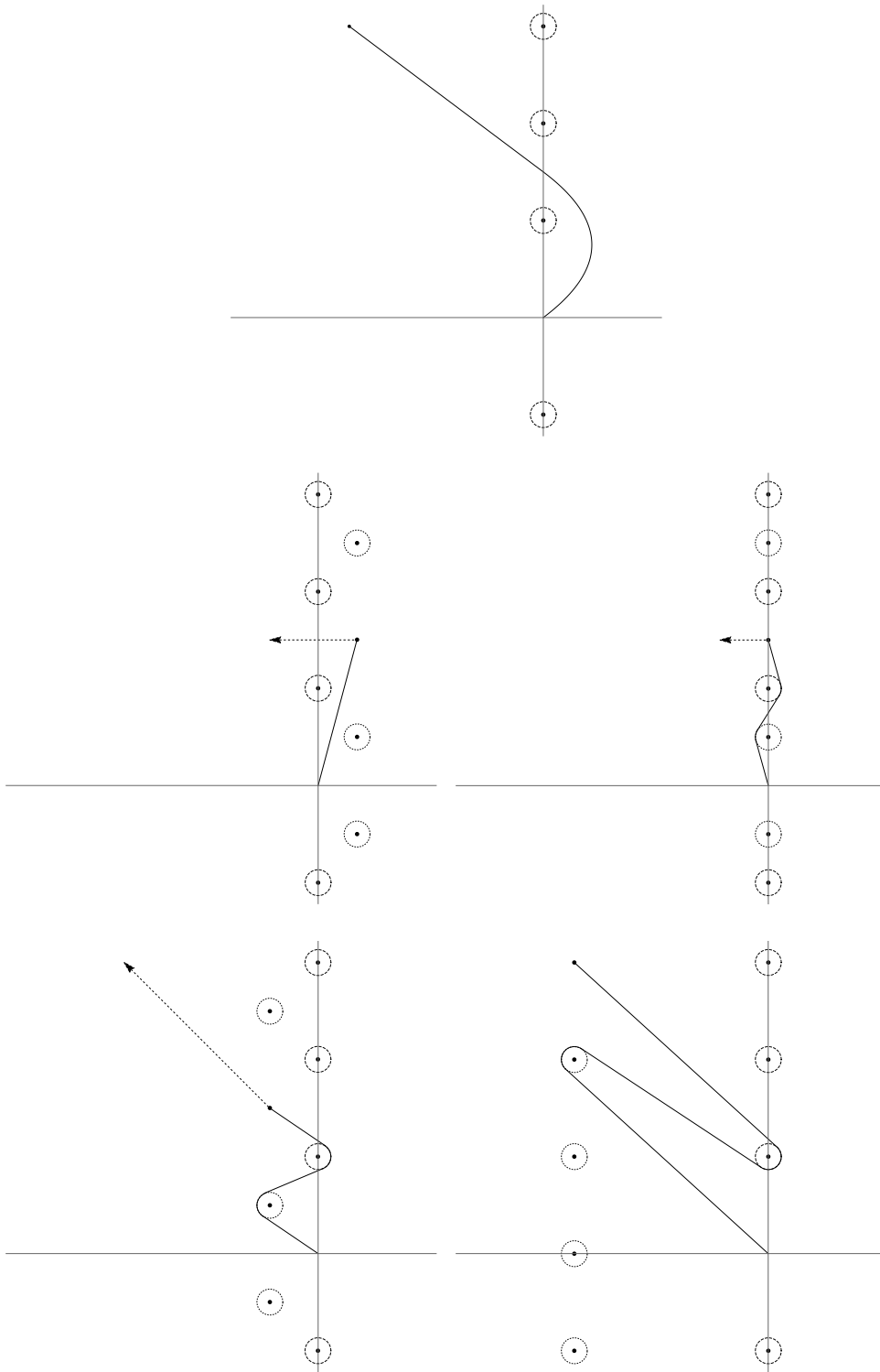


Figure 2.9: The non-symmetric initial path is shown at the top, three different steps to get the symmetric path follow with the actual symmetric path at the bottom right.

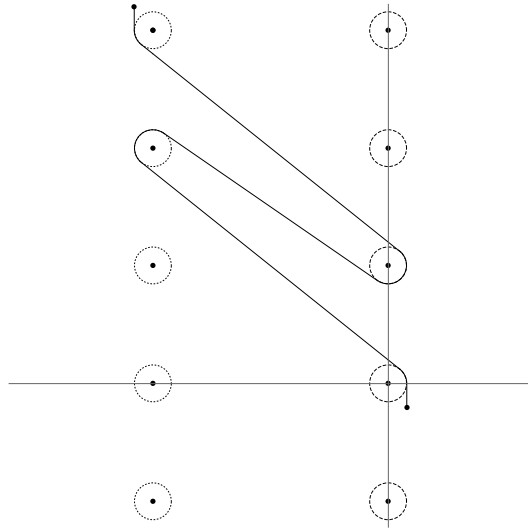


Figure 2.10: Symmetric path on \mathcal{R}_1 .

the elastic band has formed a symmetric path between λ and $\lambda e^{\pi i} + s$. See Figure 2.10 for one example using the same starting path as in Figure 2.9. However there is a slight problem with this path: it is not an element of \mathcal{R}_1 , since it originates at λ . This can be fixed because the choice of 1 as a base point of \mathcal{R}_1 was arbitrary.⁵

Construction of symmetric paths on \mathcal{R}_0

The above construction of symmetric paths is intuitive in its principle but it becomes quickly hard to visualize. Here we will present a straightforward algorithm that constructs the same paths.

We will describe paths whose homotopy class has the first pass between singularities happening at some s with $\text{Im } s > 2\pi$ and happens from right to left. We will denote these passings by a finite sequence $N \in \mathbb{N}^* \times (\mathbb{Z}^*)^m$, with m depending on the sequence considered. Notice that the first term of this sequence has to be positive. We write this sequence as $\langle N_1, N_2, \dots, N_{m+1} \rangle$. To give meaning to this sequence we start with the path that is the straight line connecting 0 to $s_0 = 2\pi + \pi i$. We will call this *the $\langle 0 \rangle$ path*. Then we move s_0 to $s_0 + 2\pi N_1 i$ and then we cross to $s_1 = s_0 + 2\pi N_1 i - 4\pi$. In general we have $s_{n+1} = s_n + 2\pi N_n i + (-1)^n 4\pi$, with each passing through the imaginary axis happening horizontally. We will call this path *the skeleton of the path $\langle N_1, N_2, \dots, N_{m+1} \rangle$* .

⁵This does not imply that we need to move the base point of \mathcal{R}_1 . We just need to ask that λ is not a negative real number and to consider the path that is the straight line between 1 and λ and use this with the symmetric path to get an element of \mathcal{R}_1 .

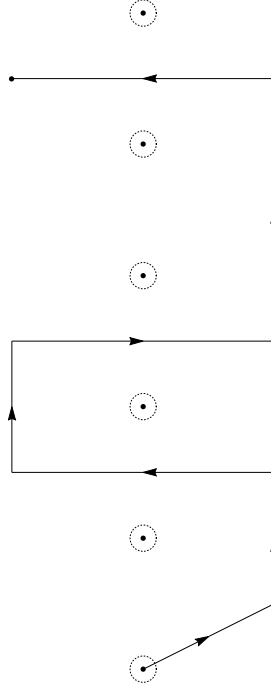
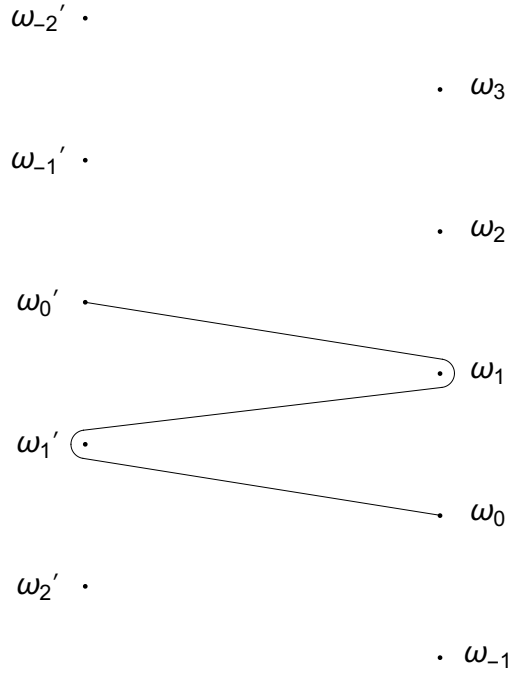


Figure 2.11: The skeleton of the path $\langle 1, 1, 2 \rangle$.

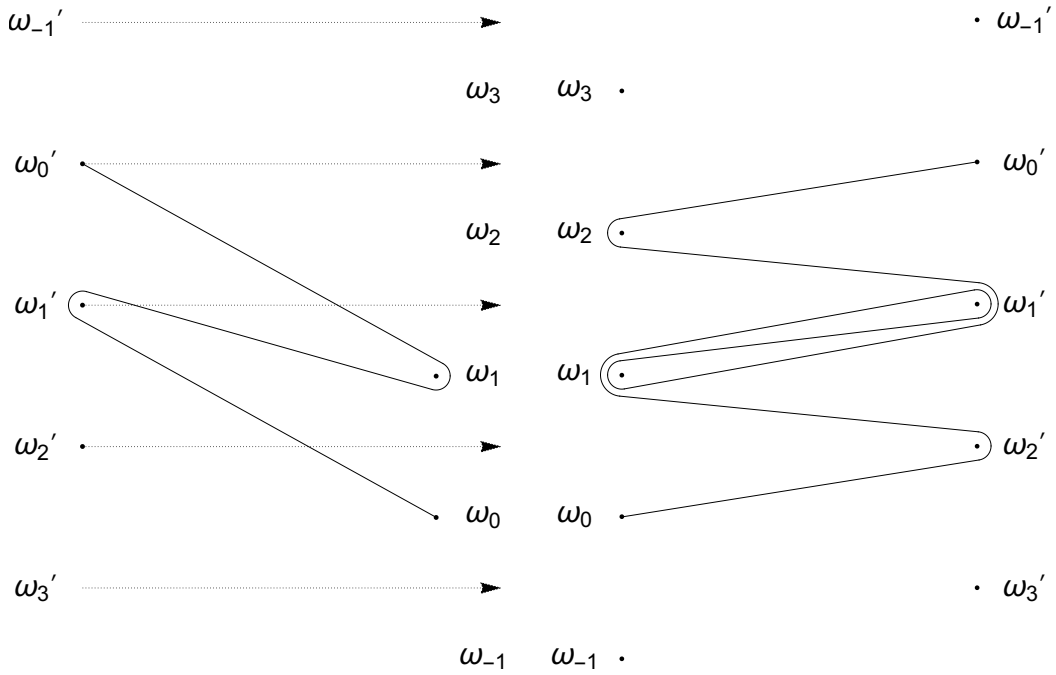
As an example we use the path $\langle 1, 1, 2 \rangle$. Its skeleton can be seen in Figure 2.11. This path can be constructed in three steps, each step is shown as a sequence of three graphs shown in Figures 2.12, 2.13 and 2.14. The immobile singularities are labeled ω_i and the mobile singularities are labeled ω'_i . From this example we can deduct right away that all paths can be made such that they do not self intersect and that they can be constructed by arcs of circles and line segments, called just *lines* from now on. Moreover, the lines are always between mobile and immobile singularities. We define the degree of ω'_i or ω_i to be the number of lines attached to them. Because the path is symmetric the degrees of ω'_i and ω_i are equal.

To find the algorithm for the construction of symmetric paths we need to calculate how many lines we get in each step. We start with the $\langle 0 \rangle$ path that has just one line and nothing else. When we look at last figure of 2.12 we see that ω'_0 has degree 1 and ω'_1 has degree 2. This happens because ω'_1 ‘jumps over’ ω_0 as we move the ending point up by 2π and as ω_0 has degree 1, ω'_1 ‘catches’ one line as it crosses and this creates 2 new lines attached to ω'_1 . On the other hand, ω'_0 ‘jumps over’ ω_1 which had degree 0 so it gains none after crossing. The dotted arrows help to visualize this.

This hints to a rule that each ω'_1 gains twice the degrees that the singularities that



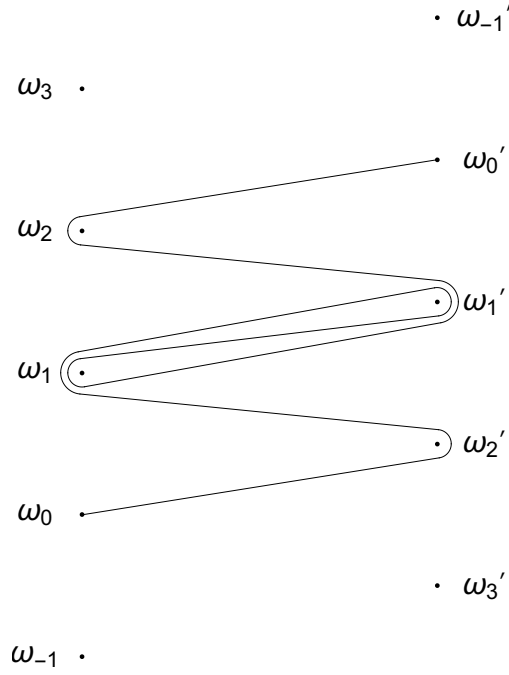
(a) We begin with the $\langle 1 \rangle$ path.



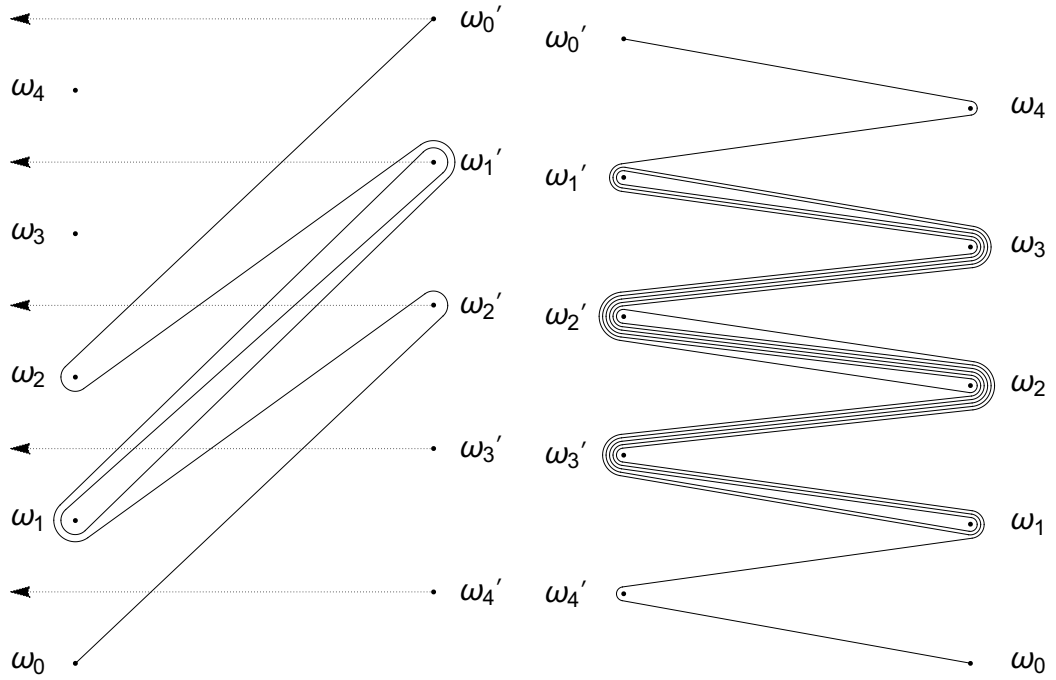
(b) We move the ending point up by 2π .

(c) We cross to the right.

Figure 2.13: Second step. The construction of the $\langle 1, 1 \rangle$ path from the $\langle 1 \rangle$ path.



(a) We begin with the $\langle 1, 1 \rangle$ path.



(b) We move the ending point up by 4π .

(c) We cross to the left.

Figure 2.14: Third step. The construction of the $\langle 1, 1, 2 \rangle$ path from the $\langle 1, 1 \rangle$ path.

jumps over had. Looking at the second step and Figure 2.13 we see that this is not exactly true. Looking at ω'_1 we see that it had degree 2 and ‘jumps over’ ω_1 which has degree 2 so we would expect it to have degree 6 in the end but it has degree 4. This happens because when we look at the number of lines it ‘catches’ as it crosses we need to take into consideration that ω'_1 and ω_1 have a common line. This would normally imply that we need to keep track of all the lines between all mobile and immobile singularities. However we see that each time we cross each ω'_i can be connected only to two ω_i ’s. So we only need to keep track for each ω'_i how many lines connect upwards and how many connect downwards. We will call these numbers *upwards* and *downwards degree of ω'_i* respectively. Of course the degree of ω'_i is the sum of these two.

Let $\mathbf{N} = \langle N_1, N_2, \dots, N_m \rangle$ and let $\mathbf{N}_n = \langle N_1, N_2, \dots, N_n \rangle$ for any $n \in \{1, 2, \dots, m\}$. We denote $\deg_{\mathbf{N}_n}^\uparrow(i)$ the upwards degree of ω'_i just after the n -th step.⁶ Similarly we denote $\deg_{\mathbf{N}_n}^\downarrow(i)$ the downwards degree and $\deg_{\mathbf{N}_n}(i)$ the degree of ω'_i . To get the degrees of the path \mathbf{N} we start by setting $\deg_{\{0\}}^\uparrow(i) = \mathbf{1}_{\{0\}}(i)$, $\deg_{\{0\}}^\downarrow(i) = 0$ for all $i \in \mathbb{Z}$ and $\kappa_1 = 0$.

Then for every $n \in \{1, 2, \dots, m\}$ we do the following steps:

1. For all $i \in \mathbb{Z}$:
 - If $N_n > 0$, then $J_i = \{\kappa_n - i + 1, \kappa_n - i + 2, \dots, \kappa_n - i + N_n\}$.
 - If $N_n < 0$, then $J_i = \{\kappa_n - i, \kappa_n - i - 1, \dots, \kappa_n - i + N_n + 1\}$.
2. For all $i \in \mathbb{Z}$:
 - If $N_n > 0$ then $\deg_{\mathbf{N}_n}^\uparrow(i) = \sum_{j \in J_i} \deg_{\mathbf{N}_{n-1}}(j) - \deg_{\mathbf{N}_{n-1}}^\uparrow(i)$
and $\deg_{\mathbf{N}_n}^\downarrow(i) = \sum_{j \in J_i} \deg_{\mathbf{N}_{n-1}}^\downarrow(j) + \deg_{\mathbf{N}_{n-1}}^\downarrow(i)$.
 - If $N_n < 0$ then $\deg_{\mathbf{N}_n}^\uparrow(i) = \sum_{j \in J_i} \deg_{\mathbf{N}_{n-1}}(j) + \deg_{\mathbf{N}_{n-1}}^\uparrow(i)$
and $\deg_{\mathbf{N}_n}^\downarrow(i) = \sum_{j \in J_i} \deg_{\mathbf{N}_{n-1}}^\downarrow(j) - \deg_{\mathbf{N}_{n-1}}^\downarrow(i)$.
3. We set $\kappa_{n+1} = \kappa_n + N_n$.

By doing this for the path $\langle 1, 1, 2 \rangle$ we get $\deg_3(0) = 1$, $\deg_3(1) = 6$, $\deg_3(2) = 12$, $\deg_3(3) = 10$, $\deg_3(4) = 2$ and every other degree is 0, as expected. The arcs of the paths are drawn having different radii. This is not a requirement of the process, it is just a way to help to visualize the path. In practice the path can be defined with all arcs having the same radii.

Notice that the path can be drawn such that it does not self intersect. This means that by knowing the degrees of all singularities we can reconstruct the path. In

⁶In other words the upwards degree of ω'_i of the path \mathbf{N}_n .

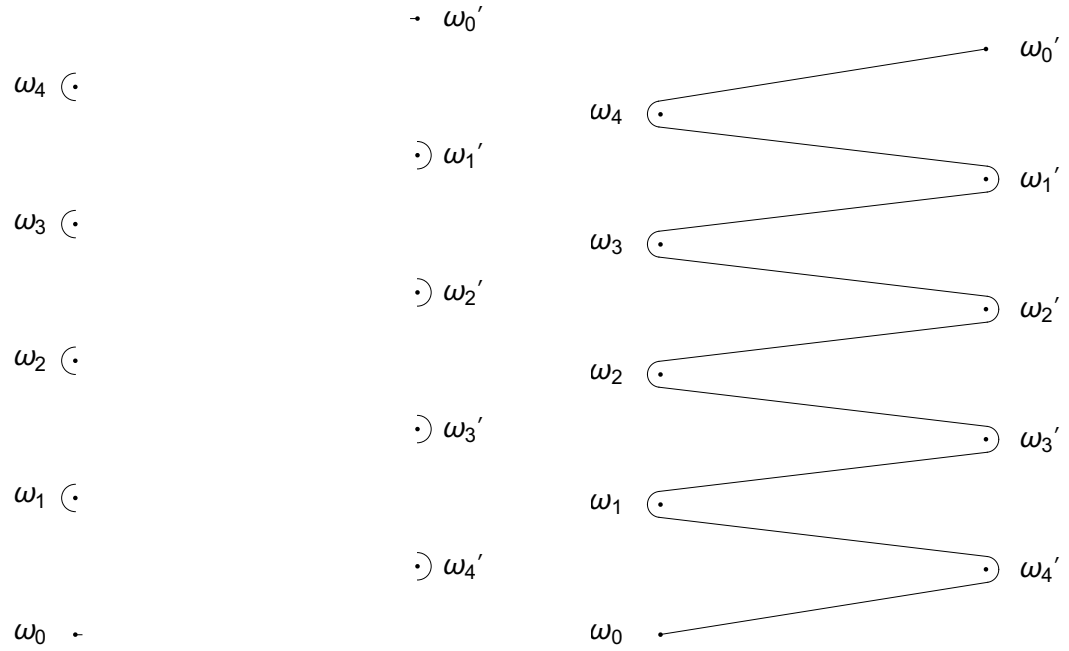


Figure 2.15: On the left we see the trace of the $\langle 4 \rangle$ path and on the right the path itself.

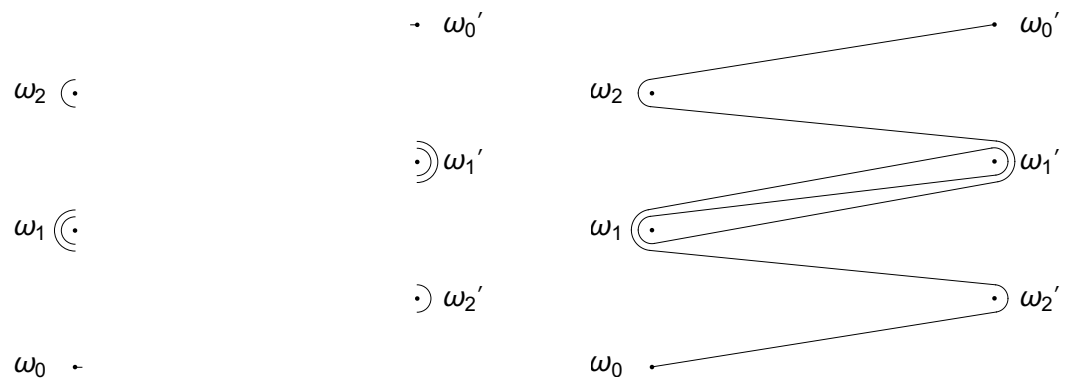


Figure 2.16: On the left we see the trace of the $\langle 1, 1 \rangle$ path and on the right the path itself.

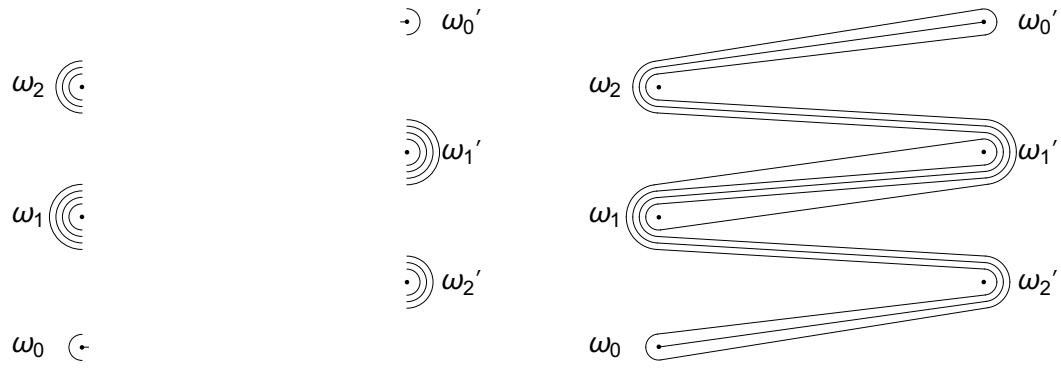


Figure 2.17: On the left we see the trace of the $\langle 1, -1, 2 \rangle$ path and on the right the path itself.

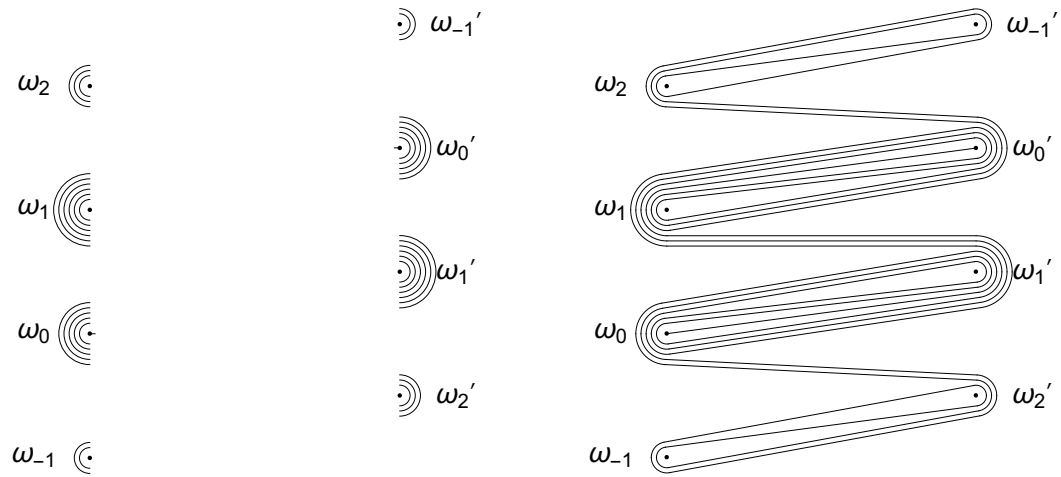


Figure 2.18: On the left we see the trace of the $\langle 2, -2, 1 \rangle$ path and on the right the path itself.

order to do this we draw the *trace* of the path, that is just the arcs of the path, or in other words it is the path without its lines. This can be done easily since all ω'_i with $i \neq 0$ have even degrees and ω'_0 has odd. So we draw $\lfloor \deg(i)/2 \rfloor$ semicircles at each ω'_i keeping in mind that there is a line that connects directly to ω'_0 . Then since we know that the path can be drawn without self intersections we connect the edges of the semicircles (plus ω'_0) in sequence from top to bottom.

We give some examples of this construction in Figures 2.15, 2.16, 2.17 and 2.18. The $\langle 4 \rangle$ path, for which we have $\deg_{\langle 4 \rangle}(0) = 1$, $\deg_{\langle 4 \rangle}(1) = \deg_{\langle 4 \rangle}(2) = \deg_{\langle 4 \rangle}(3) = \deg_{\langle 4 \rangle}(4) = 2$ and everything else is 0, is shown in Figure 2.15. For the $\langle 1, 1 \rangle$ path we have $\deg_{\langle 1, 1 \rangle}(0) = 1$, $\deg_{\langle 1, 1 \rangle}(1) = 4$, $\deg_{\langle 1, 1 \rangle}(2) = 2$ and everything else 0. For the $\langle 1, -1, 2 \rangle$ path we have $\deg_{\langle 1, -1, 2 \rangle}(0) = 3$, $\deg_{\langle 1, -1, 2 \rangle}(1) = 8$, $\deg_{\langle 1, -1, 2 \rangle}(2) = 6$ and everything else 0. And finally for the $\langle 2, -2, 1 \rangle$ path we have $\deg_{\langle 2, -2, 1 \rangle}(-1) = 4$, $\deg_{\langle 2, -2, 1 \rangle}(0) = 11$, $\deg_{\langle 2, -2, 1 \rangle}(1) = 12$, $\deg_{\langle 2, -2, 1 \rangle}(2) = 6$ and everything else 0.

We define the degree of a symmetric path γ by $\deg(\gamma) = \sum_{n \in \mathbb{Z}} \deg_\gamma(n)$. Given any path γ of \mathcal{R}_0 with this algorithm we can draw a symmetric path in the same homotopy class. If the first crossing of γ is from left to right, we reflect the path around the imaginary axis, we construct the symmetric path and we reflect it again. If the first crossing happens on the lower half-plane, we reflect the path around the real axis. Finally once we have a symmetric path with the same crossings as γ it is trivial to move the ending point to the ending point of γ . Through this we can extend the definition of \deg in \mathcal{R}_0 by setting the degree of a path to be the degree of a symmetric path in the same homotopy class.

We denote the process of constructing a symmetric path given an arbitrary path γ by the operator sym and we write $\text{sym}(\gamma)$ for the symmetric path. Notice that the operator sym is ambiguous since we have not chosen the radii of the semicircles of the path. This will be done in the next section.

Partial ordering of paths

The goal of this section is to show that the path $\text{sym} \circ P_t(\gamma)$ cannot have larger degree than the path $\text{sym}(\gamma)$. To this end we need to keep track⁷ of $P_t \circ H_\tau$ as τ changes from 0 to 1 and in order to do that we need to extend, just for this section, the notation we used to describe the crossings. We write $\mathbf{N} = \langle (N_1, b_1), \dots, (N_m, b_m) \rangle$ with N_i 's taking the same values as before plus 0 and $b_i \in \mathbb{Z}_2$. In the new extended notation the path $\langle N_1, \dots, N_m \rangle$ is written as $\langle (N_1, 1), \dots, (N_m, 1) \rangle$.

⁷ Recall that H is the homotopy used in the proof of the existence of symmetric paths.

In the extended notation the first number in the pair (N_i, b_i) denotes, as before, the vertical movement and the second number denotes whether there was a crossing or not. The path $\langle 0 \rangle$ is now denoted by $\langle (0, 0) \rangle$. Using this we can define a partial ordering of paths.

Definition 2.25. Let $\mathbf{M} = \langle (M_1, b_1), \dots, (M_n, b_n) \rangle$ and $\mathbf{N} = \langle (N_1, \beta_1), \dots, (N_n, \beta_n) \rangle$. We write

$$\mathbf{M} \preceq \mathbf{N}$$

if for all $i \in \{1, \dots, n\}$, $\text{sign}(M_i) = \text{sign}(N_i)$ ⁸, $|M_i| \leq |N_i|$ and $b_i \leq \beta_i$.

Lemma 2.26. For any symmetric path γ and any $t \in [0, 1]$ it holds $\text{sym} \circ P_t(\gamma) \preceq \gamma$.

Proof. In order to check the relation we need to see what happens when the path crosses the imaginary axis. Let $\mathbf{N} = \langle (N_1, 1), \dots, (N_n, 1), (N_{n+1}, 1) \rangle$. We assume that for any $t \in [0, 1]$ it holds $\text{sym}(\mathbf{N}_n^t) \preceq \mathbf{N}_n$. We fix t and we look at what the point⁹ $\mathbf{N}_n(t)$ does when we do the final crossing N_{n+1} . We have two cases for the vertical movement.

- Assume that $N_{n+1} > 0$. Then the endpoint of the path moves upwards by $2\pi N_{n+1}$ and then crosses. We see that the point $\mathbf{N}_n(t)$ can move only upwards and at most a distance of $2\pi N_{n+1}$.
- Similarly when $N_{n+1} < 0$ the point $\mathbf{N}_n(t)$ moves downwards a distance at most $2\pi |N_{n+1}|$ before it crosses.

We also get two cases for the crossing; namely the point either crosses or not.

If $\mathbf{N}_n(t)$, after moving vertically, crosses on the same level, between the same singularities, we denote this by $(0, 1)$. If $\mathbf{N}_n(t)$ after moving vertically by N'_i crosses we denote this by $(N'_i, 1)$ and if it does not cross after this vertical movement we denote this by $(N'_i, 0)$.

Then the result can be proved by induction since for the path $\langle (0, 0) \rangle$, for any $t \in [0, 1]$, it holds trivially that $\text{sym}(\langle (0, 0) \rangle^t) \preceq \langle (0, 0) \rangle$. \square

Lemma 2.27. If $\mathbf{M} \preceq \mathbf{N}$ then $\deg(\mathbf{M}) \leq \deg(\mathbf{N})$.

Proof. In order to compare the degrees of two paths written in the extended notation we have to translate that to the previous notation. This can be done by following

⁸ For the purpose of this definition we assume that for all $n \in \mathbb{Z}$ the relation $\text{sign}(n) = \text{sign}(0)$ is true.

⁹ Note that the point is defined relatively to the length of the path.

the directives that the extended notation encodes and keeping track of the actual passings between the singularities. In other words we need to remove the 0's. This is done by applying the following rules:

- $\langle (N_1, b_1), \dots, (N_{i-1}, b_{i-1}), (N_i, 0), (N_{i+1}, b_{i+1}), \dots, (N_m, b_m) \rangle$ becomes $\langle (N_1, b_1), \dots, (N_{i-1}, b_{i-1}), (N_i + N_{i+1}, b_{i+1}), \dots, (N_m, b_m) \rangle$,
- $\langle (N_1, b_1), \dots, (N_{i-1}, 1), (0, 1), (N_{i+1}, b_{i+1}), \dots, (N_m, b_m) \rangle$ becomes $\langle (N_1, b_1), \dots, (N_{i-1} + N_{i+1}, b_{i+1}), \dots, (N_m, b_m) \rangle$,
- $\langle (N_1, b_1), \dots, (N_m, b_m), (N_{m+1}, 0) \rangle$ becomes $\langle (N_1, b_1), \dots, (N_m, b_m) \rangle$,
- $\langle (N_1, b_1), \dots, (N_{m-1}, b_{m-1}), (N_m, 1), (0, 1) \rangle$ becomes $\langle (N_1, b_1), \dots, (N_{m-1}, b_{m-1}) \rangle$,
- $\langle (0, b_1), (N_2, b_2), \dots, (N_m, b_m) \rangle$ becomes $\langle (N_2, b_2), \dots, (N_m, b_m) \rangle$,
- $\langle (N_1, 0), (N_2, b_2), \dots, (N_m, b_m) \rangle$ becomes $\langle (N_1 + N_2, b_2), \dots, (N_m, b_m) \rangle$.

We apply these rules starting from the first pair and moving towards the last. In the end we are left with $\langle (N_1, 1), \dots, (N_n, 1) \rangle$ which we rewrite as $\langle N_1, \dots, N_n \rangle$.

However, following this procedure we may have that $N_1 < 0$. In this case we transform $\langle N_1, N_2, \dots, N_n \rangle$ to $\langle -N_1 - 1, -N_2, \dots, -N_n \rangle$. If $-N_1 - 1 = 0$ we remove it and apply the same rule to $\langle -N_2, \dots, -N_n \rangle$ again if necessary. \square

The above two lemmas prove the following.

Lemma 2.28. *For any path $\gamma \in \mathcal{R}_0$ it is true that $\deg(\text{sym} \circ P_t(\gamma)) \leq \deg(\gamma)$.*

2.5.8 Bounds for convolution

We saw earlier the definition of convolution, i.e. the convolution of f and g is

$$f * g(s) = \int_0^s f(t)g(s-t)dt.$$

We have a straightforward way to bound this convolution.

Lemma 2.29. *Let f, g be analytic in a neighbourhood V of the origin and let F, G be non decreasing, non negative, continuous functions of \mathbb{R}^+ such that for all $s \in V$ it holds that $|f(s)| \leq F(|s|)$ and $|g(s)| \leq G(|s|)$. Then*

$$|f * g(s)| \leq F * G(|s|).$$

Proof.

$$\begin{aligned}
|f * g(s)| &\leq \left| \int_0^s f(t)g(s-t)dt \right| \\
&= \left| \int_0^1 f(st)g(s(1-t))sdt \right| \\
&\leq \int_0^1 |f(st)| |g(s(1-t))| |s|dt \\
&\leq \int_0^1 F(|s|t) G(|s|(1-t)) |s|dt \\
&\leq \int_0^{|s|} F(t) G(|s|-t)dt \\
&= F * G(|s|). \quad \square
\end{aligned}$$

As we will see, it is important to have a definition of the convolution that works also with function that have branching singularities. For this we need to consider the convolution of two functions over a symmetric path.

Definition 2.30. Let $\gamma : [0, 1] \rightarrow \mathbb{C}$. We say that a function f is analytic on γ if there exists $\varepsilon > 0$ and a finite set $T = \{t_0, t_1, \dots, t_n\} \subset [0, 1]$ such that $\gamma \subset \cup_{t \in T} \mathbb{D}_\varepsilon(\gamma(t))$, $\mathbb{D}_\varepsilon(\gamma(t_{i-1})) \cap \mathbb{D}_\varepsilon(\gamma(t_i))$ is non empty and f is analytic on $\mathbb{D}_\varepsilon(\gamma(t_{i-1})) \cup \mathbb{D}_\varepsilon(\gamma(t_i))$ for all $i \in \{1, 2, \dots, n\}$.

If f is analytic on γ then we can write $f(\gamma)$ to denote the value of the analytic continuation of f to the point $\gamma(1)$. In other words, around each point on γ there is an open neighbourhood on which f is analytic so we can extend uniquely f along γ and $f(\gamma)$ is the value of this extension at the point $\gamma(1)$. Of course if f has no branching then $f(\gamma)$ does not depend on the path but just on the point $\gamma(1)$. In the general case $f(\gamma)$ does actually depend on the path, or more precisely on the homotopy class of the path.

A first approach to define the convolution along a symmetric path γ could be

$$f * g(\gamma) = \int_0^1 f(\gamma(t)) g(\gamma(1-t)) \dot{\gamma}(t) dt.$$

However this definition is ambiguous since $\gamma(t), \gamma(1-t) \in \mathbb{C}$ so $f(\gamma(t))$ and $g(\gamma(1-t))$ are not defined uniquely. To get over this difficulty we write $f(\gamma^t)$ and $g(\gamma^{1-t})$ instead. This leads to the following definition.

Definition 2.31. Let γ be a symmetric path and let f and g be analytic on γ . We define

$$f * g(\gamma) = \int_0^1 f(\gamma^t) g(\gamma^{1-t}) \dot{\gamma}(t) dt.$$

This definition is compatible with the one given at the beginning of this section for non-branching functions. This can be checked by taking the path $\gamma(t) = st$, which is clearly symmetric, and $\dot{\gamma}(t) = s$.

A lemma to bound this generalized convolution, similar to Lemma 2.29, will be proved useful. Such lemma is possible but it requires a careful construction of a special type of symmetric paths, that will be defined in the next section.

Bounds for convolution on \mathcal{R}_0

In this section we derive a generalization of Lemma 2.29. An obstacle for this generalization is that we define convolution using a symmetric path γ but then we need to bound γ^t which is in general non symmetric. If we try to solve the problem by taking the symmetric path homotopic to γ^t , we get a path that may be longer than γ^t itself. To resolve this we need to define a special operator that takes any path as an argument and gives a homotopic symmetric path of at most equal length as a result.

Let $\varepsilon < 1$, $\mathbb{C}_\varepsilon^* = \mathbb{C} \setminus (\cap_{n \in \mathbb{Z}} \mathbb{D}_\varepsilon(2\pi ni))$ and $\mathbb{C}_\varepsilon = \mathbb{C} \setminus (\cap_{n \in \mathbb{Z}^*} \mathbb{D}_\varepsilon(2\pi ni))$, and consider the set of continous and piecewise continuously differentiable functions $\gamma : [0, 1] \rightarrow \mathbb{C}_\varepsilon$ such that $\gamma(0) = 0$, $|\dot{\gamma}(t)| = |\gamma|$ and $\gamma : [\frac{\varepsilon}{|\gamma|}, 1] \rightarrow \mathbb{C}_\varepsilon^*$. In other words these are paths that start at 0 with a straight line segment from 0 to the boundary of $\mathbb{D}_\varepsilon(0)$ and then it stays at a distance of at least ε from the points $2\pi i\mathbb{Z}$. We denote the set of all these paths by R_ε . Notice that this is not the set of homotopy classes.

In Section 2.5.7 we saw a way to construct symmetric paths. In that section the paths were drawn with no self-intersections, which means that we choose arcs of circles with unequal radii. Here we denote the process described in 2.5.7 but with arcs of radius ε by $\overline{\text{sym}}_\varepsilon$. Notice that $\overline{\text{sym}}_\varepsilon : R_\varepsilon \rightarrow R_\varepsilon$. As examples, we can see in the left figure of 2.19 the path $\overline{\text{sym}}_\varepsilon\langle 4 \rangle$, in the middle figure of 2.20 the path $\overline{\text{sym}}_\varepsilon\langle 1, 1 \rangle$ and in the middle figure of 2.21 the path $\overline{\text{sym}}_\varepsilon\langle 1, -1, 2 \rangle$.

Our goal is to define an operator $\text{sym}_\varepsilon : R_\varepsilon \rightarrow R_\varepsilon$ such that the length of the path $\text{sym}_\varepsilon \circ P_t \circ \text{sym}_\varepsilon[\gamma]$ is shorter than the length of the path $P_t \circ \text{sym}_\varepsilon[\gamma]$.

Let us check that the operator $\overline{\text{sym}}_\varepsilon$ does not satisfy this requirement. We take the $\langle 4 \rangle$ path as an example, see Figure 2.19. On the left figure of 2.19 the path $\overline{\text{sym}}_\varepsilon \langle 4 \rangle$ is depicted. We have that $\max_{t \in [0,1]} \text{Re}(\overline{\text{sym}}_\varepsilon \langle 4 \rangle(t)) = \text{Re}(\overline{\text{sym}}_\varepsilon \langle 4 \rangle(1)) + \varepsilon$. So when we take the path $\text{sym}_\varepsilon \circ P_t \circ \text{sym}_\varepsilon \langle 4 \rangle$ we have that $|\text{sym}_\varepsilon \circ P_t \circ \text{sym}_\varepsilon \langle 4 \rangle| \leq |P_t \circ \text{sym}_\varepsilon \langle 4 \rangle|$ as long as $\text{Re}(\overline{\text{sym}}_\varepsilon \langle 4 \rangle(t)) \leq \text{Re}(\overline{\text{sym}}_\varepsilon \langle 4 \rangle(1)) - \varepsilon$. The problem appears when for example the ending point of $P_t \circ \text{sym}_\varepsilon \langle 4 \rangle$ is on the right side of ω'_3 . Then the path $\overline{\text{sym}}_\varepsilon \circ P_t \circ \overline{\text{sym}}_\varepsilon \langle 4 \rangle$ extends more to the right of ω'_4 than the path $P_t \circ \overline{\text{sym}}_\varepsilon \langle 4 \rangle$.

The solution proposed in [GS01] for paths with one crossing was to construct a path which at ω_n has radius $n\varepsilon$. This is shown in Figure 2.19. This way when $P_t \circ \text{sym}_\varepsilon \langle 4 \rangle$ ends on right of ω'_3 we can symmetrize and still get smaller path. We can generalize this principle to more complicated paths, just by noticing that the rule is to increase the radius of the arc progressively for each ω_n in the order they are visited. We define sym_ε to be the map that gives these paths.

As an example we look at the $\langle 1, 1 \rangle$ path, see Figure 2.20. We see that the path goes around the singularities in the following order $\omega_0, \omega'_2, \omega_1, \omega'_1, \omega_1, \omega'_1, \omega_2$ and ω'_0 . We look at the order it visits ω_n 's which is $\omega_0, \omega_1, \omega_1$ and ω_2 , so we start with radius 0¹⁰ for ω_0 and then it goes around ω_1 with radius ε , then again around ω_1 with radius 2ε and finally around ω_2 with radius 3ε . Another example of a more complicated path can be seen in Figure 2.21.

The map sym_ε has the property we required but it has a problem: it cannot be defined on the whole R_ε . This restriction appears since as the passings of a path increase, the path consists of more arcs of increasing radius, which means that these arcs will eventually enter an ε -neighbourhood of a singularity. Because of this we define

$$\check{R}_\varepsilon = \{\gamma \in R_\varepsilon : \text{sym}_\varepsilon \gamma \subset \mathbb{C}_\varepsilon\}$$

and we arrive to the following lemma.

Lemma 2.32. *For any $0 < \varepsilon < 1$ and any $\gamma \in \check{R}_\varepsilon$ it holds*

$$|\text{sym}_\varepsilon \circ P_t \circ \text{sym}_\varepsilon[\gamma]| \leq |P_t \circ \text{sym}_\varepsilon[\gamma]|$$

for all $t \in [0, 1]$.

Proof. We know that $\deg(\text{sym}_\varepsilon \circ P_t \circ \text{sym}_\varepsilon[\gamma]) \leq \deg(P_t \circ \text{sym}_\varepsilon[\gamma])$, then by the above construction we always have smaller or at most equal radii of arcs in $\text{sym}_\varepsilon \circ P_t \circ \text{sym}_\varepsilon[\gamma]$ and this ensures that the length of the path is smaller. \square

¹⁰This means that the line connects directly to the point.

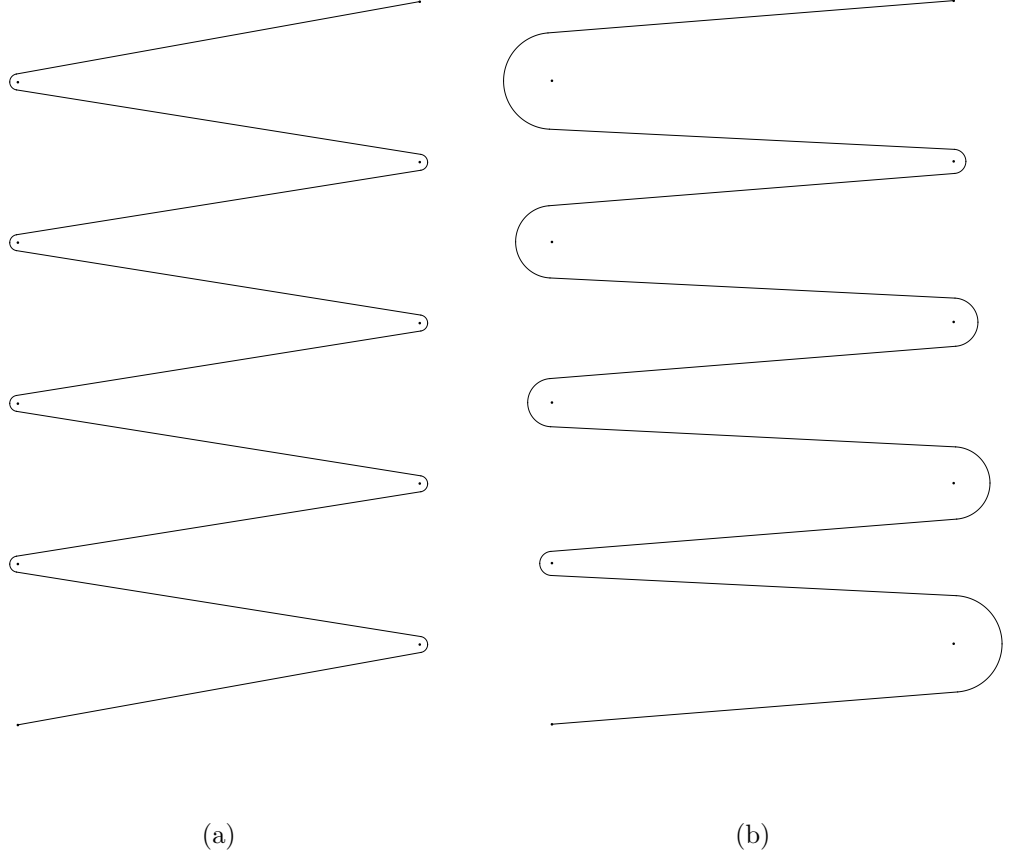


Figure 2.19: The paths $\overline{\text{sym}}_\varepsilon \langle 4 \rangle$ and $\text{sym}_\varepsilon \langle 4 \rangle$ can be seen in the left and right figures respectively.

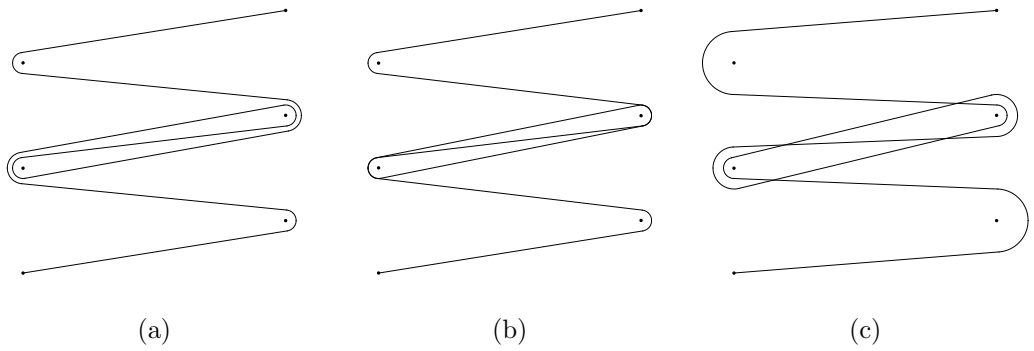


Figure 2.20: The paths $\langle 1, 1 \rangle$, $\overline{\text{sym}}_\varepsilon \langle 1, 1 \rangle$ and $\text{sym}_\varepsilon \langle 1, 1 \rangle$ can be seen in the left, centre and right figures respective.

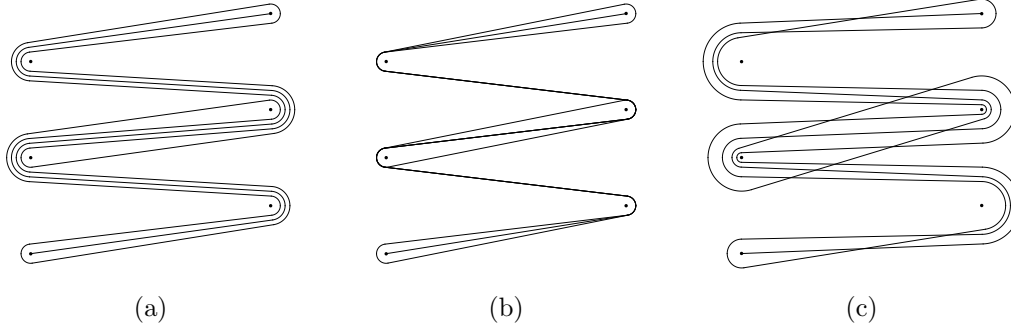


Figure 2.21: The paths $\langle 1, -1, 2 \rangle$, $\overline{\text{sym}}_\varepsilon \langle 1, -1, 2 \rangle$ and $\text{sym}_\varepsilon \langle 1, -1, 2 \rangle$ can be seen in the left, centre and right figures respective.

Notice that for each path γ the largest radius of a semicircle of the path $\text{sym}_\varepsilon \gamma$ is $(\deg(\gamma) - 1)\varepsilon$. This implies that if $\gamma \in \check{R}_\varepsilon$ then¹¹ $(\deg(\gamma) - 1)\varepsilon < \pi$ and $\text{dist}(\gamma(1), 2\pi i\mathbb{Z}) > \deg(\gamma)\varepsilon$.

This means that with given ε there is a limit to the number of crossings a path in \check{R}_ε can have. We also have that $\check{R}_{\varepsilon_1} \subset \check{R}_{\varepsilon_2}$ if $\varepsilon_2 \leq \varepsilon_1$. We define \mathcal{R}_ε to be the set of all homotopy classes of \check{R}_ε . Notice that if $\varepsilon_n \rightarrow 0$ monotonically then the $\mathcal{R}_{\varepsilon_n}$'s form an increasing sequence whose limit is \mathcal{R}_0 .

As a shorthand notation we define $\check{\gamma}_\varepsilon := \text{sym}_\varepsilon \gamma$ and $\check{\gamma}_\varepsilon^t := P_t \circ \text{sym}_\varepsilon \gamma$. We note that for any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$ and any path $\gamma \in \check{R}_\varepsilon$ if $\pi(\gamma)$ is sent to infinity along the half line $e^{i\theta}\mathbb{R}^+$ then $|\check{\gamma}| \approx \deg(\gamma)|\pi(\gamma)|$. In other words we see that the length of the path $\check{\gamma}$ has a linear growth with respect to the growth of the endpoint of the path γ .

We define the operator $\text{shrt}_\varepsilon : R_\varepsilon \rightarrow R_\varepsilon$ by choosing the shortest path in the same homotopy class. An example of this can be seen in Figure 2.22. We define the shorthand notation $\bar{\gamma}_\varepsilon := \text{shrt}_\varepsilon \gamma$ and $\bar{\gamma}_\varepsilon^t := P_t \circ \text{shrt}_\varepsilon \gamma$.

In order to get bounds we will need later, we need to restrict the ways that paths can approach infinity. We define $\mathbb{C}_{\varepsilon,n} = \mathbb{C}_\varepsilon \setminus (\mathbb{R}^+ \cdot \mathbb{D}_\varepsilon(\omega_{\pm n}))$, i.e. $\mathbb{C}_{\varepsilon,n}$ is just \mathbb{C}_ε where we remove what is shadowed by the ε -neighbourhoods of ω_n and ω_{-n} . See Figure 2.23 for an example. Then for $s \in \mathbb{C}_{\varepsilon,n}$ with $|s| > 2\pi n$ we have $2\pi n|\text{Re}(s)| \geq \varepsilon|s|$.

We define

$$R_{\varepsilon,n,\Lambda} := \{\gamma \in \check{R}_\varepsilon : \check{\gamma}_\varepsilon \subset \mathbb{C}_{\varepsilon,n}, |\bar{\gamma}_\varepsilon| \leq \Lambda|\pi(\gamma)|\}$$

and we denote by $\mathcal{R}_{\varepsilon,n,\Lambda}$ the set of the homotopy classes of $R_{\varepsilon,n,\Lambda}$ with the usual topology. The space $\mathcal{R}_{\varepsilon,n,\Lambda}$ is actually a subset of \mathcal{R}_0 . Since \mathbb{D}_ε is defined to be

¹¹ Notice that the converse is not true as these are not sufficient conditions.

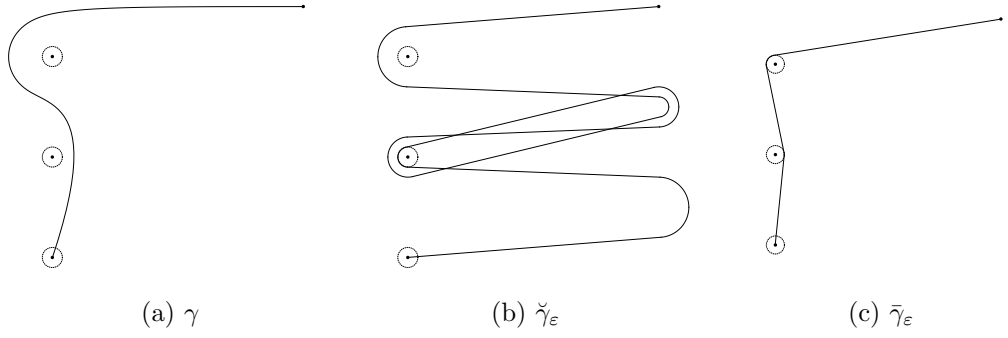


Figure 2.22: In the left figure we see a path $\gamma \in R_\varepsilon$. In the middle figure we see the path $\check{\gamma}_\varepsilon$. In the right figure we see the path $\bar{\gamma}_\varepsilon$.

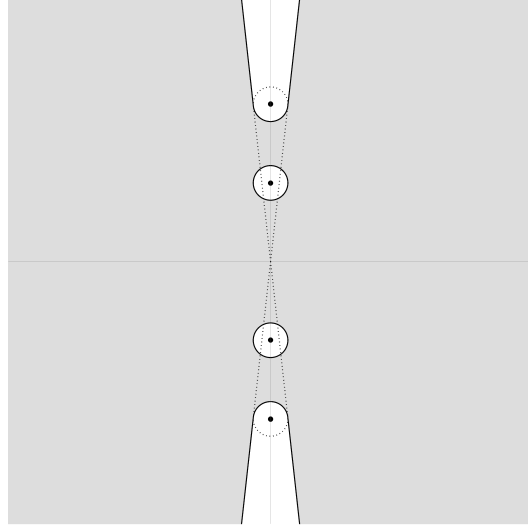


Figure 2.23: The domain $\mathbb{C}_{\varepsilon,2}$.

open, $\mathcal{R}_{\varepsilon,n,\Lambda}$ is closed and its boundary can be defined¹². If $\{\varepsilon_n\}_{n \geq 0} \rightarrow 0$ and $\{\Lambda_n\}_{n \geq 0} \rightarrow \infty$ monotonically then $\mathcal{R}_{\varepsilon_n,n,\Lambda_n}$ define an increasing sequence whose limit is \mathcal{R}_0 . Notice that both operators sym_ε and shrt_ε can be thought of acting on $\mathcal{R}_{\varepsilon,n,\Lambda}$ in a straightforward way.

We define

$$\begin{aligned} \mathcal{R}_{\varepsilon,n,\Lambda}^+ &:= \text{cl} \{ \gamma \in \mathcal{R}_{\varepsilon,n,\Lambda} : \text{Re}(\pi(\gamma)) \geq -1 \text{ and if } |\gamma| > 2\pi n \text{ then } \text{Re}(\pi(\gamma)) \geq 0 \}, \\ \mathcal{R}_{\varepsilon,n,\Lambda}^- &:= \text{cl} \{ \gamma \in \mathcal{R}_{\varepsilon,n,\Lambda} : \text{Re}(\pi(\gamma)) \leq 1 \text{ and if } |\gamma| > 2\pi n \text{ then } \text{Re}(\pi(\gamma)) \leq 0 \} \end{aligned}$$

and finally

$$\mathcal{R}_{\varepsilon,n,\Lambda}^\pm(k) := \{ \gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}^\pm : |\bar{\gamma}| \leq k \}.$$

¹²We can describe its boundary locally using the projection π .

Notice that for increasing k the sets $\mathcal{R}_{\varepsilon,n,\Lambda}^{\pm}(k)$ define a compact exhaustion of $\mathcal{R}_{\varepsilon,n,\Lambda}^{\pm}$.

Let $C^{\omega}(\mathcal{R}_{\varepsilon,n,\Lambda}^{-})$ be the space of functions analytic on $\mathcal{R}_{\varepsilon,n,\Lambda}^{-}$ and continuous on its boundary. We define for $k \in \mathbb{N}$ the family of seminorms

$$\|f\|_{\mathcal{R}_{\varepsilon,n,\Lambda}^{-}(k)} = \sup_{\gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}^{-}(k)} |f(\gamma)|.$$

This turns $C^{\omega}(\mathcal{R}_{\varepsilon,n,\Lambda}^{-})$ into a Fréchet space. We denote by $C^{\omega}(\mathcal{R}_{\varepsilon,n,\Lambda}^{-})^2$ the direct product of two such spaces and we give it the family of seminorms

$$\left\| \begin{pmatrix} f_1(\gamma) \\ f_2(\gamma) \end{pmatrix} \right\|_{\times 2, \mathcal{R}_{\varepsilon,n,\Lambda}^{-}(k)} = \max\{\|f_1\|_{\mathcal{R}_{\varepsilon,n,\Lambda}^{-}(k)}, \|f_2\|_{\mathcal{R}_{\varepsilon,n,\Lambda}^{-}(k)}\}.$$

Finally we can prove the following lemma.

Lemma 2.33. *Let $\varepsilon > 0$, $f, g \in C^{\omega}(\mathcal{R}_{\varepsilon,n,\Lambda}^{-})$ and F, G be non negative, non decreasing, continuous functions on \mathbb{R}^+ . If for all $\gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}^{-}$ it holds that $|f(\gamma)| \leq F(|\check{\gamma}_{\varepsilon}|)$ and $|g(\gamma)| \leq G(|\check{\gamma}_{\varepsilon}|)$ then*

$$|f * g(\gamma)| \leq F * G(|\check{\gamma}_{\varepsilon}|).$$

Proof. We have $f * g(\gamma) = \int_0^1 f(\check{\gamma}_{\varepsilon}^t) g(\check{\gamma}_{\varepsilon}^{1-t}) \dot{\check{\gamma}}_{\varepsilon}(t) dt$, so

$$\begin{aligned} |f * g(\gamma)| &= \left| \int_0^1 f(\check{\gamma}_{\varepsilon}^t) g(\check{\gamma}_{\varepsilon}^{1-t}) \dot{\check{\gamma}}_{\varepsilon}(t) dt \right| \\ &= \left| \int_0^1 f(\text{sym}_{\varepsilon} \check{\gamma}_{\varepsilon}^t) g(\text{sym}_{\varepsilon} \check{\gamma}_{\varepsilon}^{1-t}) \dot{\check{\gamma}}_{\varepsilon}(t) dt \right| \\ &\leq \int_0^1 |f(\text{sym}_{\varepsilon} \check{\gamma}_{\varepsilon}^t)| |g(\text{sym}_{\varepsilon} \check{\gamma}_{\varepsilon}^{1-t})| |\dot{\check{\gamma}}_{\varepsilon}(t)| dt \\ &\leq \int_0^1 F(|\text{sym}_{\varepsilon} \check{\gamma}_{\varepsilon}^t|) G(|\text{sym}_{\varepsilon} \check{\gamma}_{\varepsilon}^{1-t}|) |\dot{\check{\gamma}}_{\varepsilon}| dt \\ &\leq \int_0^1 F(|\check{\gamma}_{\varepsilon}^t|) G(|\check{\gamma}_{\varepsilon}^{1-t}|) |\dot{\check{\gamma}}_{\varepsilon}| dt \\ &= \int_0^1 F(|\check{\gamma}_{\varepsilon}|t) G(|\check{\gamma}_{\varepsilon}|(1-t)) |\dot{\check{\gamma}}_{\varepsilon}| dt \\ &= F * G(|\check{\gamma}_{\varepsilon}|) \end{aligned}$$

Since by $\check{\gamma}_{\varepsilon}$ we denote the path $t \mapsto \check{\gamma}_{\varepsilon}(t)$, by $\dot{\check{\gamma}}_{\varepsilon}$ we denote the derivative of this function. By the usual convention we have $|\dot{\check{\gamma}}_{\varepsilon}(t)| = |\dot{\gamma}_{\varepsilon}|$. \square

Convolution of majorants

The following lemmas will be useful later to get proper bounds.

Lemma 2.34. *Let F, G be non-negative, continuous functions on \mathbb{R}^+ and let $\alpha \geq \beta \geq 0$. Then for all $s \in \mathbb{R}^+$ we have $(e^{\alpha s} F) * (e^{\beta s} G)(s) \leq e^{\alpha s} (F * G)(s)$.*

Proof.

$$\begin{aligned}
 (e^{\alpha s} F) * (e^{\beta s} G)(s) &= \int_0^s e^{\alpha t} F(t) e^{\beta(s-t)} G(s-t) dt \\
 &= \int_0^s e^{\alpha s} F(t) e^{(\beta-\alpha)(s-t)} G(s-t) dt \\
 &\leq \int_0^s e^{\alpha s} F(t) G(s-t) dt \\
 &= e^{\alpha s} \int_0^s F(t) G(s-t) dt \\
 &= e^{\alpha s} (F * G)(s)
 \end{aligned}
 \quad \square$$

Lemma 2.35. *Let F, G be non-negative, continuous functions on \mathbb{R}^+ . Then for all $s \in \mathbb{R}^+$ we have $(sF) * G(s) \leq s(F * G)(s)$.*

Proof. Let $s \in \mathbb{R}^+$ and F, G as above. Then we have

$$\begin{aligned}
 (sF) * G(s) &= \int_0^s t F(t) G(s-t) dt \\
 &= \int_0^1 s t F(st) G(s-st) s dt \\
 &\leq \int_0^1 s F(st) G(s-st) s dt \\
 &= s \int_0^s F(t) G(s-t) dt \\
 &= s (F * G)(s)
 \end{aligned}
 \quad \square$$

Lemma 2.36. *Let F, G be non-decreasing, non-negative, continuously differentiable functions on \mathbb{R}^+ . Then $F * G$ is also a non-decreasing, non-negative, continuously differentiable function on \mathbb{R}^+ .*

Proof. We have $F * G(0) = 0$. Let $s \geq 0$. Then

$$\frac{d}{ds} F * G(s) = F(s)G(0) + \int_0^s F(t)G'(s-t) dt.$$

Since G is non-decreasing, $G'(s)$ is non-negative on \mathbb{R}^+ , which implies that $\frac{d}{ds}F * G(s)$ is non-negative, so $F * G$ is a non-decreasing, non negative function on \mathbb{R}^+ . \square

Lemma 2.37. *Let $g \in \mathcal{B}_n(\mathbb{D}_r)$ and $h \in \mathcal{B}_m(\mathbb{D}_r)$ with $n, m \in \mathbb{N}$. Then $g * h \in \mathcal{B}_{m+n+1}(\mathbb{D}_r)$*

Proof.

$$\begin{aligned}
|g * h(s)| &= \left| \int_0^s g(t)h(s-t)dt \right| \\
&= \left| \int_0^1 g(st)h(s(1-t))sdt \right| \\
&\leq \int_0^1 |g(st)||h(s(1-t))|sdt \\
&\leq |s|^{m+n+1} \|g\|_n \|h\|_m \int_0^1 t^n(1-t)^m dt \\
&= |s|^{m+n+1} \|g\|_n \|h\|_m B(n+1, m+1) \\
&= |s|^{m+n+1} \|g\|_n \|h\|_m \frac{m!n!}{(m+n+1)!}.
\end{aligned}$$

Thus $\|g * h\|_{m+n+1} \leq \|g\|_n \|h\|_m \frac{m!n!}{(m+n+1)!}$, which implies that $g * h \in \mathcal{B}_{m+n+1}(\mathbb{D}_r)$. \square

Corollary 2.38. *Let $u \in \mathcal{B}_0(\mathbb{D}_r)$. Then $\|u^{*n}\|_{n-1} \leq \frac{\|u\|_0^n}{(n-1)!}$.*

Lemma 2.39. *For all $s, \alpha > 0$ it is true that*

$$\frac{s^n}{n!} e^{-\alpha s} \leq \alpha^{-n}.$$

Proof. We will use the fact that $n!e^n \geq n^n$. Let $g(s) := s^n e^{-\alpha s}$. First we show that

$$\max_{s>0} g(s) = n^n (\alpha e)^{-n}.$$

For fixed n we have $g(s) \geq 0$, $g(0) = \lim_{s \rightarrow \infty} g(s) = 0$, $g(s)$ is continuous on \mathbb{R}^+ and $g'(s) = s^{n-1} e^{-\alpha s} (n - \alpha s)$. So the maximum is indeed $n^n (\alpha e)^{-n}$. This gives

$$\frac{g(s)}{n!} \leq \frac{g(s)e^n}{n^n} \leq \frac{n^n e^n}{\alpha^n e^n n^n} = \alpha^{-n},$$

which completes the proof. \square

Chapter 3

Splitting of separatrices of an area-preserving map at 3:1 resonance

Let F_0 be a map as described in Section 1.3 for which the coefficient b_0 does not vanish and F_0 agrees at least up to order 4 with the normal form, Proposition 1.4. For this section we drop the subscript and denote this map simply by F . By the assumption we have ¹

$$F(x, y) = \begin{pmatrix} F_x(x, y) \\ F_y(x, y) \end{pmatrix} := \begin{pmatrix} x - 2b_0xy + b_0^2x^3 + b_0^2xy^2 + g_x(x, y) \\ y - b_0x^2 + b_0y^2 + b_0^2x^2y + b_0^2y^3 + g_y(x, y) \end{pmatrix},$$

with

$$F_x(x, y) := \sum_{n \geq 0} \sum_{m=0}^n \frac{f_{x|n,m}}{m!(n-m)!} x^m y^{n-m}$$

¹ By the assumption we know the map up to order 4 so in practice we have

$$\begin{aligned} g_x(x, y) &= -8(72b_0^3 + b_1)x^3y - 24b_1xy^3 + O_5(x, y), \\ g_y(x, y) &= -2(36b_0^3 + 5b_1)x^4 - 12x^2y^2(36b_0^3 - b_1)x^2y^2 + 6(36b_0^3 + b_1)y^4 + O_5(x, y). \end{aligned}$$

However this is used only in section 3.3 to prove the existence of the fundamental solution to the variational equation.

and similar for $F_y(x, y)$, with g_x and g_y having quadruple root at the origin. From standard Cauchy estimates² we get the existence of constants $M, a > 0$ such that

$$\max\{|f_{x|n,m}|, |f_{y|n,m}|\} \leq Mm!(n-m)!a^n. \quad (3.1)$$

By the implicit function theorem we get the existence of the inverse map in a neighbourhood of the origin and without loss of generality we can assume that both are analytic in the same neighbourhood. The inverse map is of the form

$$F^{-1}(x, y) := \begin{pmatrix} x + 2b_0xy + b_0^2x^3 + b_0^2xy^2 + h_x(x, y) \\ y + b_0x^2 - b_0y^2 + b_0^2x^2y + b_0^2y^3 + h_y(x, y) \end{pmatrix} \quad (3.2)$$

with the same estimates.

We consider the equation

$$W(t+1) = F(W(t)). \quad (3.3)$$

We search for a solution of this equation such that $W(t)$ is analytic around infinity and $\lim_{t \rightarrow \infty} W(t) = 0$. Notice that if such solution exists it can be represented as a Laurent series around infinity. In the general case this solution does not exist. However, we will see that a formal solution of this form, \mathscr{W} , always exists and it is Borel-Laplace summable. The results of the present chapter are summarized in the next theorem.

Theorem 3.1. *There exists a unique formal solution with real coefficients,*

$$\mathscr{W}(t) = \begin{pmatrix} 0 \\ -\frac{1}{b_0 t} \end{pmatrix} + O(|t|^{-3}) \in \frac{1}{t} \mathbb{C}[[\frac{1}{t}]]^2,$$

of equation (3.3) and any other formal solution of the form $\mathscr{W}'(t) = (0, -\frac{1}{b_0 t}) + O(|t|^{-2})$ can be written as $\mathscr{W}(t+c)$ for some $c \in \mathbb{C}$. Moreover there exists a formal solution with real coefficients, $\tilde{\Xi} \in t^2 \mathbb{C}[[\frac{1}{t}]]^2$, of the equation

$$X(t+1) = F'(\mathscr{W}(t)) \cdot X(t),$$

² If we deal with a family of maps that are at 1:3 resonance then we have to assume that the family is analytic in an open neighbourhood of the origin and that it is uniformly bounded in the same neighbourhood. The constant a is the inverse of the radius of analyticity of the map and M depends on the supremum, so under these assumptions the estimates are true.

such that

$$\tilde{\Xi}(t) = \begin{pmatrix} b_0 t^2 - 18 \frac{b_1}{b_0^2} + \frac{24b_1^2}{b_0^5} t^{-2} \\ -\frac{8a_0}{b_0^3} t^{-1} \end{pmatrix} + O(|t|^{-3}),$$

and $\det(\tilde{\Xi}(t), \dot{\mathcal{W}}(t)) = 1$.

The Borel transform of \mathcal{W} is a function, \hat{W} , analytic on \mathcal{R}_0 and is of exponential growth along any path that crosses the imaginary axis finitely many time and does not go to infinity vertically.

The Borel-Laplace summation of \mathcal{W} gives two solutions of the equation (3.3), W^+ and W^- , that satisfy $\lim_{t \rightarrow \pm\infty} W^\pm(t) = 0$. There exist two complex constants, θ and ρ , such that for any $t \in \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq 1, \operatorname{Im}(z) < 0\}$, with $|t|$ big enough, it holds

$$W^+(t) - W^-(t) \asymp e^{-2\pi i t} \left(\theta \tilde{\Xi}(t) + \rho \dot{\mathcal{W}}(t) \right) + O(t^7 e^{-4\pi i t})$$

and

$$\theta = \lim_{t \rightarrow +\infty} e^{2\pi t} \omega(W^+(-it) - W^-(-it), \dot{W}^-(-it)).$$

Remark. The constant θ will be called the *Stokes constant* of the map F . It gives the size of the splitting on the transversal direction. The constant ρ gives the tangent size of the splitting. It should also be noted that there is the possibility of $\theta = 0$ and $\rho \neq 0$. Since ρ gives the tangent splitting and θ the transverse splitting, it seems strange to have only tangent splitting of separatrices, but the present analysis cannot exclude this.

Notice that in the asymptotic formula of Theorem 3.1 we include the term $O(t^7 e^{-4\pi i t})$. The constants θ and ρ depend on the map F and can both vanish. If this happens we cannot exclude the possibility that $\Delta_{\omega_2}^+ \mathcal{W}$ does not vanish, so the difference will be of order $t^7 e^{-4\pi i t}$. Of course if at least one of the two constants is non-zero then the last term is meaningless since it is exponentially smaller than the first.

If F_ν is a family of maps that depends analytically on ν and that for all ν the map F_ν is at 1:3 resonance, then the constants θ_ν and ρ_ν also depend analytically on ν . So we can conclude that for any such family we have that either θ_ν is zero for all ν or that it is non-zero almost everywhere. This extends to a family depending analytically on any finite number of parameters. We have a similar conclusion for

the constant ρ_ν .

Overview of the proof

Initially we find a formal solution, \mathscr{W} , of the equation (3.3) in the space $\frac{1}{t}\mathbb{C}[[t^{-1}]]^2$. This can be proved inductively by substituting a formal series in (3.3) and collecting terms.

Then we take the Borel transform of (3.3). Knowing \mathscr{W} , we show that the solution to this is in RES^{sim} and that it agrees with the Borel transform of \mathscr{W} for the first 2 orders. In order to find this solution we use the fact that RES^{sim} is isomorphic to $\mathbb{C}^\omega(\mathcal{R}_0)$.

As a first step we prove that the Borel transform of the equation can be turned into a contraction on $\mathbb{C}^\omega(\mathbb{D}_r)$, for r small enough, and we get a sequence that converges uniformly to the solution. Then we prove that this sequence is majorated to get that it converges uniformly on any compact subset of \mathcal{R}_0 .

Information about the singularities can be obtained by the Borel transform of the fundamental solution of the variational equation. This determines $\Delta_{\omega_1}\mathscr{W}$ up to two constants. Then we proceed inductively to get $\Delta_{\omega_n}^+\mathscr{W}$.

3.1 Asymptotics of the separatrices

Lemma 3.2. *Equation (3.3) admits a unique formal solution $\mathscr{W}(t) = (\mathscr{W}_x(t), \mathscr{W}_y(t))$, with*

$$\mathscr{W}(t) = \begin{pmatrix} \sum_{i \geq 1} W_{x,i} t^{-i} \\ \sum_{i \geq 1} W_{y,i} t^{-i} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{b_0 t} \end{pmatrix} + O(|t|^{-3}).$$

Proof. We substitute the series $\mathscr{W}(t)$ to equation (3.3) keeping in mind that

$$(t+1)^{-m} = \sum_{k \geq 0} (-1)^k \binom{m+k-1}{k} t^{-k-m}$$

and expand equation (3.3) in series of decreasing powers of t .

It turns out that the leading coefficient must be $(0, -1/b_0)$ and the rotations of this

by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. The coefficient of t^{-2} is free. Then the higher order coefficients are determined inductively. \square

The convergence of the series is not given and we will see that in general it is divergent. This implies that there cannot exist a solution analytic around infinity.

3.2 Existence of the Borel transform of the separatrices

3.2.1 Separatrix equation

Let $W(t) := (w_x(t), w_y(t))$ be a separatrix of the map, meaning that W satisfies $W(t+1) = F(W(t))$ and $\lim_{t \rightarrow +\infty} W(t) = 0$ or $\lim_{t \rightarrow -\infty} W(t) = 0$. We know that the map admits a formal separatrix of the form $\mathcal{W}(t) = (0, -\frac{1}{b_0 t}) + O(t^{-3})$, so we can search for a separatrix of the form

$$\begin{aligned} w_x(t) &:= \frac{u_x(t)}{t^2}, \\ w_y(t) &:= -\frac{1}{b_0 t} + \frac{u_y(t)}{t^2}. \end{aligned}$$

We write

$$(t+1)^2 (W(t+1) - F(W(t))) = 0$$

and since

$$\begin{aligned} F_x \left(\frac{u_x(t)}{t^2}, -\frac{1}{b_0 t} + \frac{u_y(t)}{t^2} \right) &= \sum_{n \geq 0} \sum_{m=0}^n \frac{F_x^{(m, n-m)}(0, -\frac{1}{b_0 t})}{m!(n-m)!} \frac{u_x(t)^m u_y(t)^{n-m}}{t^{2n}} \\ &= \sum_{n \geq 0} \sum_{m=0}^n \sum_{k \geq 0} \frac{(-1)^k f_{x|n+k, m}}{(b_0)^k m!(n-m)!k!} \frac{u_x(t)^m u_y(t)^{n-m}}{t^{2n+k}}, \end{aligned}$$

the equation becomes the following system of equations

$$\begin{aligned} u_x(t+1) - u_x(t) - \frac{4u_x(t)}{t} &= \left(\frac{1}{t^4} + \frac{4}{t^3} + \frac{6}{t^2} \right) u_x(t) \\ &\quad - 2b_0 \left(\frac{1}{t^5} + \frac{3}{t^4} + \frac{3}{t^3} + \frac{1}{t^2} \right) u_x(t) u_y(t) \\ &\quad + b_0^2 \left(\frac{1}{t^6} + \frac{2}{t^5} + \frac{1}{t^4} \right) u_x(t) u_y(t)^2 \end{aligned}$$

$$\begin{aligned}
& + b_0^2 \left(\frac{1}{t^6} + \frac{2}{t^5} + \frac{1}{t^4} \right) u_x(t)^3 \\
& + \left(1 + \frac{1}{t} \right)^2 \sum_{n \geq 0} \sum_{m=0}^n \sum_{\substack{k \geq 0 \\ k \geq 4-n}} \frac{(-1)^k f_{x|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{u_x(t)^m u_y(t)^{n-m}}{t^{2n+k-2}},
\end{aligned}$$

$$\begin{aligned}
u_y(t+1) - u_y(t) = & - \frac{1}{b_0} \left(\frac{1}{t^3} + \frac{1}{t^2} \right) \\
& + \left(\frac{3}{t^4} + \frac{4}{t^3} \right) u_y(t) \\
& - b_0 \left(\frac{1}{t^5} + \frac{3}{t^4} + \frac{3}{t^3} + \frac{1}{t^2} \right) u_x(t)^2 \\
& - b_0 \left(\frac{3}{t^5} + \frac{5}{t^4} + \frac{1}{t^3} - \frac{1}{t^2} \right) u_y(t)^2 \\
& + b_0^2 \left(\frac{1}{t^6} + \frac{2}{t^5} + \frac{1}{t^4} \right) u_x(t)^2 u_y(t) \\
& + b_0^2 \left(\frac{1}{t^6} + \frac{2}{t^5} + \frac{1}{t^4} \right) u_y(t)^3 \\
& + \left(1 + \frac{1}{t} \right)^2 \sum_{n \geq 0} \sum_{m=0}^n \sum_{\substack{k \geq 0 \\ k \geq 4-n}} \frac{(-1)^k f_{y|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{u_x(t)^m u_y(t)^{n-m}}{t^{2n+k-2}}.
\end{aligned}$$

Since everything was written avoiding non-negative powers of t we can use the classical definition of the Borel transform to get an element of $\mathbb{C}[[s]]$. The Borel transform of these equations gives

$$\begin{aligned}
(e^{-s} - 1) \hat{u}_x(s) - 4 * \hat{u}_x(s) = & \left(\frac{s^3}{3!} + 4 \frac{s^2}{2} + 6s \right) * \hat{u}_x(s) \\
& - 2b_0 \left(\frac{s^4}{4!} + 3 \frac{s^3}{3!} + 3 \frac{s^2}{2} + s \right) * \hat{u}_x * \hat{u}_y(s) \\
& + b_0^2 \left(\frac{s^5}{5!} + 2 \frac{s^4}{4!} + \frac{s^3}{3!} \right) * \hat{u}_x * \hat{u}_y^{*2}(s) \\
& + b_0^2 \left(\frac{s^5}{5!} + 2 \frac{s^4}{4!} + \frac{s^3}{3!} \right) * \hat{u}_x^{*3}(s) \\
& + \sum_{n \geq 0} \sum_{m=0}^n \sum_{\substack{k \geq 0 \\ k \geq 4-n}} \frac{(-1)^k f_{x|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{s^{2n+k-3}}{(2n+k-3)!} * u_x^{*m} * u_y^{*(n-m)}(s)
\end{aligned}$$

$$+ (2+s) * \sum_{n \geq 0} \sum_{m=0}^n \sum_{\substack{k \geq 0 \\ k \geq 4-n}} \frac{(-1)^k f_{x|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{s^{2n+k-3}}{(2n+k-3)!} * u_x^{*m} * u_y^{*(n-m)}(s),$$

$$\begin{aligned} (e^{-s} - 1) \hat{u}_y(s) = & -\frac{1}{b_0} \left(\frac{s^2}{2} + s \right) \\ & + \left(3 \frac{s^3}{3!} + 4 \frac{s^2}{2} \right) * \hat{u}_y(s) \\ & - b_0 \left(\frac{s^4}{4!} + 3 \frac{s^3}{3!} + 3 \frac{s^2}{2} + s \right) * \hat{u}_x^{*2}(s) \\ & - b_0 \left(3 \frac{s^4}{4!} + 5 \frac{s^3}{3!} + \frac{s^2}{2} - s \right) * \hat{u}_y^{*2}(s) \\ & + b_0^2 \left(\frac{s^5}{5!} + 2 \frac{s^4}{4!} + \frac{s^3}{3!} \right) * \hat{u}_x^{*2} * \hat{u}_y(s) \\ & + b_0^2 \left(\frac{s^5}{5!} + 2 \frac{s^4}{4!} + \frac{s^3}{3!} \right) * \hat{u}_y^{*3}(s) \\ & + \sum_{n \geq 0} \sum_{m=0}^n \sum_{\substack{k \geq 0 \\ k \geq 4-n}} \frac{(-1)^k f_{y|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{s^{2n+k-3}}{(2n+k-3)!} * u_x^{*m} * u_y^{*(n-m)}(s) \\ & + (2+s) * \sum_{n \geq 0} \sum_{m=0}^n \sum_{\substack{k \geq 0 \\ k \geq 4-n}} \frac{(-1)^k f_{y|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{s^{2n+k-3}}{(2n+k-3)!} * u_x^{*m} * u_y^{*(n-m)}(s). \end{aligned}$$

For $n \geq 4$ we define

$$\mathcal{G}_{x|n,m}(t) := \left(1 + \frac{1}{t} \right)^2 \sum_{k \geq 0} \frac{(-1)^k f_{x|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{1}{t^{2n+k-2}}.$$

We can extend this definition for $n < 4$ if we add the terms that appear due to the expansion of the known terms. So for example we have

$$\mathcal{G}_{x|0,0}(t) := \left(1 + \frac{1}{t} \right)^2 \sum_{k \geq 4} \frac{(-1)^k f_{x|k,0}}{(b_0)^k k!} \frac{1}{t^{k-2}}$$

and

$$\mathcal{G}_{x|1,1}(t) := \frac{6}{t^2} + \frac{4}{t^3} + \frac{1}{t^4} + \left(1 + \frac{1}{t} \right)^2 \sum_{k \geq 3} \frac{(-1)^k f_{x|k+1,m}}{(b_0)^k k!} \frac{1}{t^k}.$$

The Borel transform of $\mathcal{G}_{x|n,m}$ for $n \geq 4$ is

$$\begin{aligned}\hat{\mathcal{G}}_{x|n,m}(s) &:= \sum_{k \geq 0} \frac{(-1)^k f_{x|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{s^{2n+k-3}}{(2n+k-3)!} \\ &\quad + (2+s) * \sum_{k \geq 0} \frac{(-1)^k f_{x|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{s^{2n+k-3}}{(2n+k-3)!}.\end{aligned}$$

We get similar relations for $n < 4$.

We define the operator

$$\mathcal{F} : \begin{pmatrix} u(s) \\ v(s) \end{pmatrix} \mapsto \begin{pmatrix} \sum_{n \geq 1} \sum_{m=0}^n \hat{\mathcal{G}}_{x|n,m} * u^{*m} * v^{*(n-m)}(s) \\ \sum_{n \geq 1} \sum_{m=0}^n \hat{\mathcal{G}}_{y|n,m} * u^{*m} * v^{*(n-m)}(s) \end{pmatrix}. \quad (3.4)$$

We then define the linear operator

$$L : \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \mapsto \begin{pmatrix} (e^{-s} - 1)x(s) - 4 * x(s) \\ (e^{-s} - 1)y(s) \end{pmatrix}, \quad (3.5)$$

and finally

$$\mathcal{A}(s) := \begin{pmatrix} \hat{\mathcal{G}}_{x|0,0}(s) \\ \hat{\mathcal{G}}_{y|0,0}(s) \end{pmatrix}. \quad (3.6)$$

Then the transformed separatrix equation becomes

$$L \left[\hat{U} \right] (s) = \mathcal{A}(s) + \mathcal{F} \left[\hat{U} \right] (s),$$

with $\hat{U} = (\hat{u}_x, \hat{u}_y)$.

For $\hat{\mathcal{G}}_{x|n,m}$ and $\hat{\mathcal{G}}_{y|n,m}$ we get the following bounds.

Lemma 3.3. *For all $\beta > \frac{a}{b_0}$ there exists $M_\beta, \lambda_\beta > 0$ such that for all $n \in \mathbb{N}$ and all $s \in \mathbb{C}$*

$$|\hat{\mathcal{G}}_{x|n,m}(s)|, |\hat{\mathcal{G}}_{y|n,m}(s)| \leq M_\beta \lambda_\beta^n \binom{n}{m} e^{\beta|s|}.$$

Moreover for all $n \geq 4$ and all $|s| \leq 1$ it holds

$$|\hat{\mathcal{G}}_{x|n,m}(s)|, |\hat{\mathcal{G}}_{y|n,m}(s)| \leq M_a \binom{n}{m} a^n \frac{|s|^{2n-3}}{(2n-3)!} e^{\frac{a}{b_0}|s|}$$

for some $M_a > 0$, with a the same as in equation (3.1).

Proof. Let

$$\hat{g}_{x|n,m}(s) := \sum_{k \geq 0} \frac{(-1)^k f_{x|n+k,m}}{(b_0)^k m! (n-m)! k!} \frac{s^{2n+k-3}}{(2n+k-3)!}.$$

We have

$$\begin{aligned} |\hat{g}_{x|n,m}(s)| &\leq \frac{|s|^{n-4}}{(n-4)!} * \sum_{k \geq 0} \frac{|(-1)^k f_{x|n+k,m}|}{(b_0)^k m! (n-m)! k!} \frac{|s|^{n+k}}{(n+k)!} \\ &\leq \frac{|s|^{n-4}}{(n-4)!} * \sum_{k \geq 0} \frac{M a^{n+k} m! (n+k-m)!}{(b_0)^k m! (n-m)! k!} \frac{|s|^{n+k}}{(n+k)!} \\ &= \frac{|s|^{n-4}}{(n-4)!} * \left(\frac{|s|^n}{n!} \sum_{k \geq 0} \frac{M a^{n+k} m! (n+k-m)! n!}{(b_0)^k m! (n-m)! (n+k)!} \frac{|s|^k}{k!} \right) \\ &= M \frac{a^n |s|^{n-4}}{(n-4)!} * \left(\frac{|s|^n}{n!} \sum_{k \geq 0} \frac{\binom{n}{m}}{\binom{n+k}{m}} \frac{a^k}{(b_0)^k} \frac{|s|^k}{k!} \right) \\ &\leq M \binom{n}{m} \frac{a^n |s|^{n-4}}{(n-4)!} * \left(\frac{|s|^n}{n!} \sum_{k \geq 0} \frac{a^k}{(b_0)^k} \frac{|s|^k}{k!} \right) \\ &= M \binom{n}{m} \frac{a^n |s|^{n-4}}{(n-4)!} * \left(\frac{|s|^n}{n!} e^{\frac{a|s|}{b_0}} \right) \\ &\leq M \binom{n}{m} a^n \frac{|s|^{2n-3}}{(2n-3)!} e^{\frac{a|s|}{b_0}}. \end{aligned}$$

Then evidently

$$\begin{aligned} |1 * \hat{g}_{x|n,m}(s)| &\leq M \binom{n}{m} a^n \frac{|s|^{2n-2}}{(2n-2)!} e^{\frac{a|s|}{b_0}}, \\ |s * \hat{g}_{x|n,m}(s)| &\leq M \binom{n}{m} a^n \frac{|s|^{2n-1}}{(2n-1)!} e^{\frac{a|s|}{b_0}}. \end{aligned}$$

Combining these we get the second bound of the lemma.

For the first bound we have

$$\begin{aligned} |\hat{g}_{x|n,m}(s)| &\leq M \binom{n}{m} a^n \frac{|s|^{2n-3}}{(2n-3)!} e^{\frac{a|s|}{b_0}} \\ &= M \binom{n}{m} a^n \frac{|s|^{2n-3}}{(2n-3)!} e^{\left(\frac{a}{b_0} - \beta\right)|s|} e^{\beta|s|} \end{aligned}$$

$$\begin{aligned}
&\leq M \binom{n}{m} a^n \frac{|s|^{2n-3}}{(2n-3)!} e^{\left(\frac{a}{b_0}-\beta\right)|s|} e^{\beta|s|} \\
&= M \binom{n}{m} a^n \left(\beta - \frac{a}{b_0}\right)^{-2n+3} e^{\beta|s|} \\
&\leq M \binom{n}{m} \left(\beta - \frac{a}{b_0}\right)^3 \left(\frac{a}{\left(\beta - \frac{a}{b_0}\right)^2}\right)^n e^{\beta|s|} \\
&\leq M'_\beta \lambda_\beta^n \binom{n}{m} e^{\beta|s|}.
\end{aligned}$$

Also

$$\begin{aligned}
|1 * \hat{g}_{x|n,m}(s)| &\leq M \binom{n}{m} a^n 1 * \frac{|s|^{2n-3}}{(2n-3)!} e^{\frac{a|s|}{b_0}} \\
&\leq M \binom{n}{m} a^n \frac{|s|^{2n-2}}{(2n-2)!} e^{\frac{a|s|}{b_0}} \\
&\leq M \binom{n}{m} \left(\beta - \frac{a}{b_0}\right)^2 \left(\frac{a}{\left(\beta - \frac{a}{b_0}\right)^2}\right)^n e^{\beta|s|} \\
&\leq M''_\beta \lambda_\beta^n \binom{n}{m} e^{\beta|s|},
\end{aligned}$$

and

$$\begin{aligned}
|s * \hat{g}_{x|n,m}(s)| &\leq M \binom{n}{m} a^n |s| * \frac{|s|^{2n-3}}{(2n-3)!} e^{\frac{a|s|}{b_0}} \\
&\leq M \binom{n}{m} a^n \frac{|s|^{2n-1}}{(2n-1)!} e^{\frac{a|s|}{b_0}} \\
&\leq M \binom{n}{m} \left(\beta - \frac{a}{b_0}\right) \left(\frac{a}{\left(\beta - \frac{a}{b_0}\right)^2}\right)^n e^{\beta|s|} \\
&\leq M'''_\beta \lambda_\beta^n \binom{n}{m} e^{\beta|s|}.
\end{aligned}$$

So we can add these bounds to obtain the bound of the lemma. These extend straightforwardly to the cases with $n < 4$. \square

Corollary 3.4. *The functions $\hat{\mathcal{G}}_{x|n,m}$ and $\hat{\mathcal{G}}_{y|n,m}$ are entire functions of exponential type $\frac{a}{b_0}$.*

3.2.2 Inversion of the linear part

Let L be the linear operator defined in (3.5). Then the inverse operator can be found by solving the following uncoupled linear system:

$$\begin{aligned}(e^{-s} - 1)x(s) - 4 * x(s) &= X(s) \\ (e^{-s} - 1)y(s) &= Y(s).\end{aligned}$$

The inversion of the second equation is trivial. To invert the first, initially we investigate the homogeneous linear equation

$$(e^{-s} - 1)\chi'(s) - 4\chi(s) = 0.$$

Its solution is

$$\chi(s) = (1 - e^s)^{-4}.$$

Then by substituting $x(s) = (c(s)\chi(s))'$ we get

$$c'(s) = e^s(1 - e^s)^3 X(s)$$

which gives

$$x(s) = \frac{e^s}{1 - e^s} X(s) + \frac{4e^s}{(1 - e^s)^5} \int_0^s e^t (1 - e^t)^3 X(t) dt.$$

So the inverse operator is defined by

$$L^{-1} : \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} \mapsto \begin{pmatrix} \frac{e^s}{1 - e^s} X(s) + \frac{4e^s}{(1 - e^s)^5} \int_0^s e^t (1 - e^t)^3 X(t) dt \\ \frac{e^s}{1 - e^s} Y(s) \end{pmatrix}. \quad (3.7)$$

We also define the operator

$$\mathcal{F} : \begin{pmatrix} \hat{u}_x(s) \\ \hat{u}_y(s) \end{pmatrix} \mapsto L^{-1} \left[\mathcal{A} + \mathcal{F} \left[\begin{pmatrix} \hat{u}_x \\ \hat{u}_y \end{pmatrix} \right] \right] (s) \quad (3.8)$$

without specifying the space on which it acts yet. With the help of this operator the transformed separatrix equation becomes

$$\hat{U}(s) = \mathcal{F} [\hat{U}] (s). \quad (3.9)$$

Let

$$U_n = \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} := \mathcal{F}^n \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] \quad (3.10)$$

for all $n \in \mathbb{N}$. We will show that this sequence converges to the unique fixed point of the map \mathcal{F} .

3.2.3 Existence of solution in a neighbourhood of the origin

Let \mathcal{F} be defined by (3.4) and let

$$\begin{aligned} \mathcal{F}_N[\hat{v}, \hat{u}](s) &:= \begin{pmatrix} \mathcal{F}_{x|N}[\hat{v}, \hat{u}](s) \\ \mathcal{F}_{y|N}[\hat{v}, \hat{u}](s) \end{pmatrix}, \\ \mathcal{F}_{\geq N}[\hat{v}, \hat{u}](s) &:= \begin{pmatrix} \mathcal{F}_{x|\geq N}[\hat{v}, \hat{u}](s) \\ \mathcal{F}_{y|\geq N}[\hat{v}, \hat{u}](s) \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} \mathcal{F}_{j,N}[\hat{v}, \hat{u}](s) &:= \sum_{n=0}^{N-1} \sum_{m=0}^n \hat{\mathcal{G}}_{j,n,m} * u^{*m} * v^{*(n-m)}(s), \\ \mathcal{F}_{j,\geq N}[\hat{v}, \hat{u}](s) &:= \sum_{n \geq N} \sum_{m=0}^n \hat{\mathcal{G}}_{j,n,m} * u^{*m} * v^{*(n-m)}(s), \end{aligned}$$

$j \in \{x, y\}$.

Lemma 3.5. *Let $\alpha > 0$, $N \geq 4$, $r < 1$. Then for all $\hat{u}, \hat{v} \in \mathcal{B}_0(\mathbb{D}_r)$ such that $\|\hat{u}\|_0, \|\hat{v}\|_0 \leq \alpha$ it holds $\mathcal{F}_{\geq N}[\hat{v}, \hat{u}] \in \mathcal{B}_{3N-3}(\mathbb{D}_r)^2$. Moreover, there exists $M > 0$, that depends on α , a and b_0 such that*

$$\|\mathcal{F}_{\geq N}[\hat{v}, \hat{u}]\|_{\times 2, 3N-3} \leq M \frac{(2\alpha)^N a^N}{(3N-3)!}.$$

Proof. We have

$$|\mathcal{F}_{x|\geq N}[\hat{v}, \hat{u}](s)| = \left| \sum_{n \geq 0} \sum_{m=0}^{n+N} \hat{\mathcal{G}}_{x|n+N,m} * u^{*m} * v^{*(n+N-m)}(s) \right|$$

$$\begin{aligned}
&\leq \sum_{n \geq 0} \sum_{m=0}^{n+N} \left| \hat{\mathcal{G}}_{x|n+N,m} \right| * \left| u^{*m} * v^{*(n+N-m)}(s) \right| \\
&\leq \sum_{n \geq 0} \sum_{m=0}^{n+N} M_a \binom{n+N}{m} a^{n+N} \left(\frac{|s|^{2n+2N-3}}{(2n+2N-3)!} e^{\frac{a}{b_0}|s|} \right) * \left| u^{*m} * v^{*(n+N-m)}(s) \right| \\
&= \sum_{n \geq 0} M_a a^{n+N} \left(\frac{|s|^{2n+2N-3}}{(2n+2N-3)!} e^{\frac{a}{b_0}|s|} \right) * |(\hat{u} + \hat{v})^{*n+N}(s)| \\
&\leq \sum_{n \geq 0} M_a a^{n+N} \left(\frac{|s|^{2n+2N-3}}{(2n+2N-3)!} e^{\frac{a}{b_0}|s|} \right) * \frac{|s|^{n+N-1} (\|\hat{u}\|_0 + \|\hat{v}\|_0)^{n+N}}{(n+N-1)!} \\
&\leq \sum_{n \geq 0} M_a a^{n+N} (\|\hat{u}\|_0 + \|\hat{v}\|_0)^{n+N} \frac{|s|^{3n+3N-3}}{(3n+3N-3)!} e^{\frac{a}{b_0}|s|} \\
&\leq M_a a^N (2\alpha)^N \frac{|s|^{3N-3}}{(3N-3)!} e^{\frac{a}{b_0}|s|} \sum_{n \geq 0} (2\alpha)^n a^n \frac{|s|^{3n}}{n!} \frac{n! (3N-3)!}{(3n+3N-3)!} \\
&\leq M_a a^N (2\alpha)^N \frac{|s|^{3N-3}}{(3N-3)!} e^{\frac{a}{b_0}|s|} \sum_{n \geq 0} (2\alpha)^n a^n \frac{|s|^{3n}}{n!} \\
&= M_a a^N (2\alpha)^N \frac{|s|^{3N-3}}{(3N-3)!} e^{\frac{a}{b_0}|s| + 2\alpha a |s|^3} \\
&\leq M_a a^N (2\alpha)^N e^{\frac{a}{b_0}r + 2\alpha ar^3} \frac{|s|^{3N-3}}{(3N-3)!} \\
&\leq M (2\alpha)^N a^N \frac{|s|^{3N-3}}{(3N-3)!}.
\end{aligned}$$

The same can be proved for $\mathcal{F}_{y| \geq N}[\hat{v}, \hat{u}](s)$ and from this the result follows. \square

Lemma 3.6. For all $\hat{u}, \hat{v} \in \mathcal{B}_0(\mathbb{D}_r)$, it holds $\mathcal{F}[\hat{v}, \hat{u}] \in \mathcal{B}_2(\mathbb{D}_r)^2$.

Proof. Let $a = \max\{\|\hat{u}\|_0, \|\hat{v}\|_0\}$ For $m < n$, it holds

$$\|\mathcal{F}_m[\hat{v}, \hat{u}] - \mathcal{F}_n[\hat{v}, \hat{u}]\|_{\times 2,2} \leq M_\alpha \frac{(2\alpha)^{3m} a^m r^{3m-3}}{(3m-3)!}.$$

From this we see that the partial sums form a Cauchy series, thus they converge in $\mathcal{B}_2(\mathbb{D}_r)^2$. Note that $\hat{\mathcal{G}}_{x|0,0}$, $\hat{\mathcal{G}}_{x|0,0}$, $\hat{\mathcal{G}}_{x|1,1}$, $\hat{\mathcal{G}}_{x|2,1}$, $\hat{\mathcal{G}}_{y|2,0}$ and $\hat{\mathcal{G}}_{y|2,2}$ are in \mathcal{B}_1 , $\hat{\mathcal{G}}_{y|1,0}$ is in \mathcal{B}_2 and all the other $\hat{\mathcal{G}}$ in \mathcal{B}_3 . Then it is easy to check that $\mathcal{F}_{\geq 3}[\hat{v}, \hat{u}] \in \mathcal{B}_4(\mathbb{D}_r)^2$, $\mathcal{F}_{\geq 2}[\hat{v}, \hat{u}] \in \mathcal{B}_3(\mathbb{D}_r)^2$ and $\mathcal{F}[\hat{v}, \hat{u}] \in \mathcal{B}_2(\mathbb{D}_r)^2$. \square

Lemma 3.7. Let $\alpha > 0$, $r < 1$. Then for all $\hat{u}_i, \hat{v}_i \in \mathcal{B}_0(\mathbb{D}_r)$ with $\|\hat{u}_i\|_0, \|\hat{v}_i\|_0 \leq \alpha$,

$i \in \{1, 2\}$ it holds

$$\|\mathcal{F}[\hat{v}_1, \hat{u}_1] - \mathcal{F}[\hat{v}_2, \hat{u}_2]\|_{\times 2, 1} \leq 3r \left\| \begin{pmatrix} \hat{v}_1 \\ \hat{u}_1 \end{pmatrix} - \begin{pmatrix} \hat{v}_2 \\ \hat{u}_2 \end{pmatrix} \right\|_{\times 2, 0} + O(r^2).$$

Proof. We have

$$\begin{aligned} |\mathcal{F}_x[\hat{v}_1, \hat{u}_1](s) - \mathcal{F}_x[\hat{v}_2, \hat{u}_2](s)| &\leq \left| \hat{\mathcal{G}}_{x|1,1} * (\hat{v}_1 - \hat{v}_2)(s) \right| \\ &\quad + |\mathcal{F}_{x|\geq 2}[\hat{v}_1, \hat{u}_1](s) - \mathcal{F}_{x|\geq 2}[\hat{v}_2, \hat{u}_2](s)| \\ &\leq \frac{|s|^2}{2} \left\| \hat{\mathcal{G}}_{x|1,1} \right\|_1 \|\hat{v}_1 - \hat{v}_2\|_0 + |\mathcal{R}[\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2](s)| \end{aligned}$$

with $\mathcal{R}[\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2](s) = \mathcal{F}_{x|\geq 2}[\hat{v}_1, \hat{u}_1](s) - \mathcal{F}_{x|\geq 2}[\hat{v}_2, \hat{u}_2](s) \in \mathcal{B}_3(\mathbb{D}_r)$. So we obtain the estimate

$$\begin{aligned} \|\mathcal{F}_x[\hat{v}_1, \hat{u}_1] - \mathcal{F}_x[\hat{v}_2, \hat{u}_2]\|_1 &\leq \frac{r}{2} \left\| \hat{\mathcal{G}}_{x|1,1} \right\|_1 \|\hat{v}_1 - \hat{v}_2\|_0 + O(r^2) \\ &\leq \frac{r}{2} (6 + O(r)) \|\hat{v}_1 - \hat{v}_2\|_0 + O(r^2) \\ &\leq 3r \|\hat{v}_1 - \hat{v}_2\|_0 + O(r^2). \end{aligned}$$

Similarly

$$\begin{aligned} \|\mathcal{F}_y[\hat{v}_1, \hat{u}_1] - \mathcal{F}_y[\hat{v}_2, \hat{u}_2]\|_1 &\leq \frac{r}{2} \left\| \hat{\mathcal{G}}_{y|1,0} \right\|_1 \|\hat{u}_1 - \hat{u}_2\|_0 + O(r^2) \\ &\leq \frac{r}{2} (2r + O(r^2)) \|\hat{u}_1 - \hat{u}_2\|_0 + O(r^2) = O(r^2). \end{aligned}$$

Then clearly

$$\|\mathcal{F}[\hat{v}_1, \hat{u}_1] - \mathcal{F}[\hat{v}_2, \hat{u}_2]\|_{\times 2, 1} \leq 3r \left\| \begin{pmatrix} \hat{v}_1 - \hat{v}_2 \\ \hat{u}_1 - \hat{u}_2 \end{pmatrix} \right\|_{\times 2, 0} + O(r^2). \quad \square$$

Lemma 3.8. *Let L^{-1} be the operator defined in (3.7). Then $L^{-1} : \mathcal{B}_1(\mathbb{D}_r) \rightarrow \mathcal{B}_0(\mathbb{D}_r)$ and $\|L^{-1}\| \leq 5 + O(r)$.*

Proof. Let $X(s), Y(s) \in \mathcal{B}_1(\mathbb{D}_r)$. Then we have

$$\begin{aligned} \left| \frac{e^s}{1 - e^s} X(s) \right| &\leq \left| \frac{e^s}{1 - e^s} \right| |s| \|X\|_1 \\ &= (1 + O(r)) |s| \|X\|_1. \end{aligned}$$

Also

$$\begin{aligned}
\left| \frac{4e^s}{(1-e^s)^5} \int_0^s e^t (1-e^t)^3 X(t) dt \right| &= \left| \frac{4}{(1-e^s)^2} \int_0^1 e^{s(1+t)} \frac{(1-e^{st})^3}{(1-e^s)^3} X(st) s dt \right| \\
&\leq \left| \frac{4}{(1-e^s)^2} \right| \int_0^1 |e^{s(1+t)}| \left| \frac{(1-e^{st})^3}{(1-e^s)^3} \right| |X(st)| s dt \\
&\leq \left| \frac{4s^2}{(1-e^s)^2} \right| \int_0^1 |e^{s(1+t)}| \left| \frac{(1-e^{st})^3}{(1-e^s)^3} \right| t \|X\|_1 dt \\
&\leq \|X\|_1 \left| \frac{4s^2}{(1-e^s)^2} \right| \int_0^1 |e^{s(1+t)}| \left| \frac{(1-e^{st})^3}{(1-e^s)^3} \right| dt \\
&\leq \|X\|_1 (4 + O(r)) \int_0^1 (1 + tO(r))(t^3 + t^3 O(r)) dt \\
&\leq \|X\|_1 (4 + O(r)) \int_0^1 (1 + O(r))(1 + O(r)) dt \\
&\leq \|X\|_1 (4 + O(r)).
\end{aligned}$$

Then we have

$$\begin{aligned}
\left\| L^{-1} \begin{pmatrix} X \\ Y \end{pmatrix} \right\|_{\times 2,0} &= \max\{\|X\|_1 (5 + O(r)), \|Y\|_1 (1 + O(r))\} \\
&\leq \max\{\|X\|_1, \|Y\|_1\} (5 + O(r)) \\
&= \|(X, Y)\|_{\times 2,1} (5 + O(r)),
\end{aligned}$$

which implies that $\|L^{-1}\| \leq 5 + O(r)$. \square

Lemma 3.9. *There exists $r > 0$ such that $U_n \in \mathcal{B}_0(\mathbb{D}_r)^2$ for all $n \in \mathbb{N}$ and there exists $\hat{U} \in \mathcal{B}_0(\mathbb{D}_r)^2$ such that $\{U_n\}_{n \geq 0}$ converges uniformly to \hat{U} in $\mathcal{B}_0(\mathbb{D}_r)^2$ and $\hat{U} = \mathcal{F}[\hat{U}]$.*

Proof. Since \mathcal{F} acts from $\mathcal{B}_0(\mathbb{D}_r)^2$ to $\mathcal{B}_2(\mathbb{D}_r)^2$ and $\mathcal{A} \in \mathcal{B}_1(\mathbb{D}_r)^2$, the operator \mathcal{F} is a map from $\mathcal{B}_0(\mathbb{D}_r)^2$ to $\mathcal{B}_0(\mathbb{D}_r)^2$. It holds

$$\begin{aligned}
\|L^{-1}[\mathcal{A}]\|_{\times 2,0} &\leq \|L^{-1}\| \|\mathcal{A}\|_{\times 2,1} \\
&\leq (5 + O(r)) (\|\mathcal{A}\|_{\times 2,1} + O(r)) \\
&= 5 \|\mathcal{A}\|_{\times 2,1} + O(r).
\end{aligned}$$

Fix $\alpha > 0$ and define $\mathcal{V} = \{f \in \mathcal{B}_0(\mathbb{D}_r) : \|f\|_0 \leq 5 \|\mathcal{A}\|_{\times 2,1} + \alpha\}$. Let $\hat{u}_x, \hat{u}_y \in \mathcal{V}$.

From the above we obtain

$$\|\mathcal{F}[\hat{u}_x, \hat{u}_y]\|_0 \leq 5 \|\mathcal{A}\|_{\times 2,1} + O(r),$$

which can be made smaller than $5 \|\mathcal{A}\|_{\times 2,1} + \alpha$ by choosing small enough r . This implies that \mathcal{F} fixes \mathcal{V} . Finally we get

$$\|\mathcal{F}[\hat{v}_1, \hat{u}_1] - \mathcal{F}[\hat{v}_2, \hat{u}_2]\|_{\times 2,0} \leq 15r \left\| \begin{pmatrix} \hat{v}_1 - \hat{v}_2 \\ \hat{u}_1 - \hat{u}_2 \end{pmatrix} \right\|_{\times 2,0} + O(r^2).$$

This means that by choosing r small enough \mathcal{F} can be turned into a contraction. Then the contraction mapping theorem gives us the existence of a fixed point \hat{U} and since $(0,0) \in \mathcal{V}$ we have that $\{U_n\}_{n \geq 0} \rightarrow \hat{U}$ in $\mathcal{B}_0(\mathbb{D}_r)^2$. \square

3.2.4 Bounds for the linear part.

Let X be a function analytic in a neighbourhood of the origin without constant term. Then we define

$$\mathcal{I}[X](s) := \frac{4e^s}{(1-e^s)^5} \int_0^s e^t (1-e^t)^3 X(t) dt.$$

We saw that $\mathcal{I}[X]$ is analytic in the same neighbourhood of the origin.

Recall the definition of the $\mathbb{C}_{\varepsilon,n}$, $\mathcal{R}_{\varepsilon,n,\Lambda}$ and $\mathcal{R}_{\varepsilon,n,\Lambda}^-$ defined in 2.5.8, pages 76 and 77. To extend this operator on $\mathcal{R}_{\varepsilon,n,\Lambda}$ we define:

$$\mathcal{I}[X](\gamma) := \frac{4e^{\pi(\bar{\gamma})}}{(1-e^{\pi(\bar{\gamma})})^5} \int_0^1 e^{\pi(\bar{\gamma}t)} (1-e^{\pi(\bar{\gamma}t)})^3 X(\bar{\gamma}t) \dot{\bar{\gamma}}(t) dt$$

for all $\gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}$ and X with no constant term around the origin.

Lemma 3.10. *For all $\varepsilon > 0$ there exists $C > 0$ such that $\forall s \in \mathbb{C}_{\varepsilon,n}$, $\left| \frac{1}{1-e^s} \right| \leq C$.*

Proof. We define $M(z) = \frac{1}{1-z}$. M being a Möbius transformation, it maps a circle centred at 0 of radius R to a circle centred at 1 of radius $\frac{1}{R}$. From this we can deduce that in order to have $|M(e^s)| > R$ we need $|e^s - 1| < \frac{1}{R}$. This means that we can find the wanted constant C as long as there exists $\varepsilon > 0$ such that $\forall k \in \mathbb{Z}$, $|s - 2\pi k i| > \varepsilon$. \square

These lead us to the following lemma.

Lemma 3.11. *Let $\varepsilon > 0$, $\Lambda \geq 1$. Let $X \in C^\omega(\mathcal{R}_{\varepsilon,n,\Lambda}^-)$ and B be a non negative, non decreasing, continuous function B on \mathbb{R}^+ . If for all $\gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}^-$ with $|\pi(\gamma)| \geq \varepsilon$ it holds $|X(\gamma)| \leq B(|\check{\gamma}|)$ then there exists $C_\varepsilon > 0$ such that*

$$|\mathcal{I}[X](\gamma)| \leq C_\varepsilon \Lambda B(|\check{\gamma}|).$$

Proof.

$$\begin{aligned} |\mathcal{I}[X](\gamma)| &\leq \left| \frac{4e^{\pi(\gamma)}}{(1-e^{\pi(\gamma)})^5} \int_0^1 e^{\pi(\bar{\gamma}^t)} (1-e^{\pi(\bar{\gamma}^t)})^3 X(\bar{\gamma}^t) \dot{\bar{\gamma}}(t) dt \right| \\ &\leq \left| \frac{4e^{\pi(\gamma)}}{(1-e^{\pi(\gamma)})^5} \right| \int_0^1 \left| e^{\pi(\bar{\gamma}^t)} \right| \left| 1-e^{\pi(\bar{\gamma}^t)} \right|^3 |X(\bar{\gamma}^t)| |\dot{\bar{\gamma}}(t)| dt \\ &\leq |\check{\gamma}| \left| \frac{4e^{\pi(\gamma)}}{(1-e^{\pi(\gamma)})^5} \right| \int_0^1 \left| e^{\pi(\bar{\gamma}^t)} \right| \left| 1-e^{\pi(\bar{\gamma}^t)} \right|^3 |X(\bar{\gamma}^t)| dt \\ &\leq 4\Lambda \left| \frac{e^{\pi(\gamma)}\pi(\gamma)}{(1-e^{\pi(\gamma)})^5} \right| \int_0^1 \left| e^{\pi(\bar{\gamma}^t)} \right| \left| 1-e^{\pi(\bar{\gamma}^t)} \right|^3 B(|\check{\gamma}^t|) dt \\ &\leq 4\Lambda B(|\check{\gamma}|) \left| \frac{e^{\pi(\gamma)}\pi(\gamma)}{(1-e^{\pi(\gamma)})^5} \right| \int_0^1 \left| e^{\pi(\bar{\gamma}^t)} \right| \left| 1-e^{\pi(\bar{\gamma}^t)} \right|^3 dt \\ &\leq C_\varepsilon \Lambda B(|\check{\gamma}|). \end{aligned} \quad \square$$

We can extend the operator L^{-1} on $\mathcal{R}_{\varepsilon,n,\Lambda}^-$ by defining

$$L^{-1} \left[\begin{pmatrix} X \\ Y \end{pmatrix} \right] (\gamma) = \begin{pmatrix} \frac{e^{\pi(\gamma)}}{1-e^{\pi(\gamma)}} X(\gamma) + \mathcal{I}[X](\gamma) \\ \frac{e^{\pi(\gamma)}}{1-e^{\pi(\gamma)}} Y(\gamma) \end{pmatrix}.$$

Corollary 3.12. *Let $\varepsilon > 0$, $\Lambda \geq 1$, then there exists a constant $C_\varepsilon > 0$ such that if $X, Y \in C^\omega(\mathcal{R}_{\varepsilon,n,\Lambda}^-)$ and there exists a non negative, non decreasing, continuous function B on \mathbb{R}^+ such that for all $\gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}^-$ such that $|\pi(\gamma)| \geq \varepsilon$ it holds $|X(\gamma), Y(\gamma)| \leq B(|\check{\gamma}|)$ then*

$$\left| L^{-1} \left[\begin{pmatrix} X \\ Y \end{pmatrix} \right] (\gamma) \right| \leq \begin{pmatrix} C_\varepsilon(\Lambda+1)B(|\check{\gamma}|) \\ C_\varepsilon(\Lambda+1)B(|\check{\gamma}|) \end{pmatrix}$$

with the absolute value and the inequality interpreted termwise.

Proof. Direct application of Lemmas 3.10 and 3.11. \square

3.2.5 Extension of the solution towards $-\infty$

Hereon we assume that r is the radius of a neighbourhood of the origin where we already know that the solution exists and $0 < \varepsilon < r$.

In order to extend the solution we will apply Montel's theorem.

Definition 3.13. Let $D \subset \mathbb{C}$ be an open domain and let \mathcal{K} be a family of functions analytic on D . Then \mathcal{K} is called normal if every sequence in \mathcal{K} contains a subsequence that converges uniformly on all compact subsets of D .

Theorem 3.14 (Montel). *A uniformly bounded family of holomorphic functions defined on an open subset of the complex numbers is normal.*

For a proof of this theorem see [Con78].

To apply this theorem we need an auxiliary operator. Let $C, \beta, \lambda_\beta > 0$ and let $\mathcal{G}_{\beta,C} : C^\omega(\mathbb{C})^2 \rightarrow C^\omega(\mathbb{C})^2$ be defined by

$$\mathcal{G}_{\beta,C} : \begin{pmatrix} \hat{\Phi}_x(s) \\ \hat{\Phi}_y(s) \end{pmatrix} \mapsto \begin{pmatrix} \sum_{n \geq 0} \sum_{m=0}^n C \lambda_\beta^n \binom{n}{m} e^{\beta s} * \hat{\Phi}_x^{*m} * \hat{\Phi}_y^{*(n-m)}(s) \\ \sum_{n \geq 0} \sum_{m=0}^n C \lambda_\beta^n \binom{n}{m} e^{\beta s} * \hat{\Phi}_x^{*m} * \hat{\Phi}_y^{*(n-m)}(s) \end{pmatrix}.$$

Lemma 3.15. *There exists an entire function $\hat{\Phi}_{\beta,C}$ of exponential type $\beta + 8\lambda_\beta C$, non negative, non decreasing on positive reals, such that*

$$\mathcal{G}_{\beta,C} \left[\begin{pmatrix} \hat{\Phi}_{\beta,C} \\ \hat{\Phi}_{\beta,C} \end{pmatrix} \right] (s) = \begin{pmatrix} \hat{\Phi}_{\beta,C}(s) \\ \hat{\Phi}_{\beta,C}(s) \end{pmatrix}.$$

Proof. We search for a fixed point of this operator of the form $(\hat{\Phi}(s), \hat{\Phi}(s))$ with $\hat{\Phi}(s) \in C^\omega(\mathbb{C})$. This gives the equation

$$\hat{\Phi}(s) = e^{\beta s} * \sum_{n \geq 0} C \lambda_\beta^n 2^n \hat{\Phi}^{*n}(s)$$

and the Laplace transform of this equation is

$$\Phi(t) = \sum_{n \geq 0} C \lambda_\beta^n 2^n \frac{1}{t - \beta} \Phi(t)^n = \frac{C}{t - \beta} \frac{1}{1 - 2\lambda_\beta \Phi(t)}.$$

The last equation can be solved algebraically and it has two solutions. We define

$$\Phi_{\beta,C}(t) = \frac{t - \beta - \sqrt{(t - \beta)(t - \beta - 8\lambda_\beta C)}}{4\lambda_\beta(t - \beta)}.$$

Because $\lim_{|t| \rightarrow \infty} \Phi_{\beta,C}(t) = 0$, $\Phi_{\beta,C}$ is analytic outside a disk centered on the origin of radius $\beta + 8\lambda_\beta C$ and can be written as a Taylor series around infinity with positive coefficients. Then the Borel transform

$$\hat{\Phi}_{\beta,C}(s) := \mathcal{B}[\Phi_{\beta,C}](s),$$

exists and it is an entire function of exponential type $\beta + 8\lambda_\beta C$, whose restriction to \mathbb{R}^+ is a non negative, non decreasing function. \square

Lemma 3.16. *For all $\beta > \frac{a}{b_0}$, $\Lambda \geq 1$, $n \in \mathbb{N}$ there exists $C_\Lambda > 0$, such that if for $\hat{u}_x, \hat{u}_y \in C^\omega(\mathcal{R}_{\varepsilon,n,\Lambda}^-)$ it holds*

$$\left| \begin{pmatrix} \hat{u}_x(\gamma) \\ \hat{u}_y(\gamma) \end{pmatrix} \right| \leq \begin{pmatrix} \hat{\Phi}_{\beta,C}(|\check{\gamma}|) \\ \hat{\Phi}_{\beta,C}(|\check{\gamma}|) \end{pmatrix}$$

for all $\gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}^-$ then

$$\left| \mathcal{F} \begin{bmatrix} \hat{u}_x \\ \hat{u}_y \end{bmatrix} (\gamma) \right| \leq \mathcal{G}_{\beta,C} \begin{bmatrix} \hat{\Phi}_{\beta,C} \\ \hat{\Phi}_{\beta,C} \end{bmatrix} (|\check{\gamma}|) = \begin{pmatrix} \hat{\Phi}_{\beta,C}(|\check{\gamma}|) \\ \hat{\Phi}_{\beta,C}(|\check{\gamma}|) \end{pmatrix} \quad (3.11)$$

with the absolute value and inequality interpreted componentwise.

Remark. We will see that this implies that \hat{U} is of exponential type $\beta + 8\lambda_\beta C_\varepsilon(\Lambda + 1)$ on $\gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}^-$, for any $\beta > \frac{a}{b_0}$. Taking into account the definition of λ_β we find that \hat{U} is of exponential type

$$\tau(\Lambda, \varepsilon) := \frac{a}{b_0} + 3\sqrt[3]{2} \sqrt[3]{aC_\varepsilon(\Lambda + 1)}.$$

Proof. Since $|\hat{u}_x(\gamma)| \leq \hat{\Phi}_{\beta,C}(|\check{\gamma}|)$ and $|\hat{u}_y(\gamma)| \leq \hat{\Phi}_{\beta,C}(|\check{\gamma}|)$, it holds

$$\left| \hat{\mathcal{G}}_{x|n,m} * \hat{u}_x^{*m} * \hat{u}_y^{*(n-m)}(\gamma) \right| \leq M_\beta \lambda_\beta^n \binom{n}{m} \left(e^{\beta|\check{\gamma}|} * \hat{\Phi}_{\beta,C}^{*n}(|\check{\gamma}|) \right).$$

Then there exists $C > 0$ such that

$$\left| L^{-1} \begin{bmatrix} \hat{\mathcal{G}}_{x|n,m} * \hat{u}_x^{*m} * \hat{u}_y^{*(n-m)} \\ \hat{\mathcal{G}}_{y|n,m} * \hat{u}_x^{*m} * \hat{u}_y^{*(n-m)} \end{bmatrix} (\gamma) \right| \leq \begin{pmatrix} C(\Lambda + 1) \lambda_\beta^n \binom{n}{m} e^{\beta|\check{\gamma}|} * \hat{\Phi}_{\beta,C}^{*n}(|\check{\gamma}|) \\ C(\Lambda + 1) \lambda_\beta^n \binom{n}{m} e^{\beta|\check{\gamma}|} * \hat{\Phi}_{\beta,C}^{*n}(|\check{\gamma}|) \end{pmatrix}$$

so

$$\begin{aligned} \left| \mathcal{F} \begin{bmatrix} \hat{u}_x \\ \hat{u}_y \end{bmatrix} (\gamma) \right| &\leq \begin{pmatrix} \sum_{n \geq 0} \sum_{m=0}^n C(\Lambda + 1) \lambda_\beta^n \binom{n}{m} e^{\beta|\check{\gamma}|} * \hat{\Phi}_{\beta,C}^{*n}(|\check{\gamma}|) \\ \sum_{n \geq 0} \sum_{m=0}^n C(\Lambda + 1) \lambda_\beta^n \binom{n}{m} e^{\beta|\check{\gamma}|} * \hat{\Phi}_{\beta,C}^{*n}(|\check{\gamma}|) \end{pmatrix} \\ &\leq \mathcal{G}_{\beta,C_\Lambda} \begin{bmatrix} \hat{\Phi}_{\beta,C_\Lambda} \\ \hat{\Phi}_{\beta,C_\Lambda} \end{bmatrix} (|\check{\gamma}|) = \begin{pmatrix} \hat{\Phi}_{\beta,C_\Lambda}(|\check{\gamma}|) \\ \hat{\Phi}_{\beta,C_\Lambda}(|\check{\gamma}|) \end{pmatrix}, \end{aligned}$$

with $C_\Lambda = C(\Lambda + 1)$. Again, the absolute value and the inequality should be interpreted componentwise. \square

Lemma 3.17. *Let $\{U_n\}_{n \geq 0} \subset C^\omega(\mathcal{R}_{\varepsilon,n,\Lambda}^-)$ be defined by (3.10). Then for all $\beta > \frac{a}{b_0}$, $\Lambda \geq 1$, $n \in \mathbb{N}$, there exists $C_{\varepsilon,\Lambda} > 0$, such that for all $\gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}^-$*

$$|U_n(\gamma)| \leq \begin{pmatrix} \hat{\Phi}_{\beta,C_\Lambda}(|\check{\gamma}|) \\ \hat{\Phi}_{\beta,C_\Lambda}(|\check{\gamma}|) \end{pmatrix},$$

the absolute value and the inequality should be interpreted componentwise.

Proof. Since $U_0 = (0, 0)$, the result follows trivially from Lemma 3.16. \square

Lemma 3.18. *For all $\beta > \frac{a}{b_0}$, $\Lambda \geq 1$, $n \in \mathbb{N}$ it holds $U_m \rightarrow \hat{U}$ in $C^\omega(\mathcal{R}_{\varepsilon,n,\Lambda}^-)$.*

Proof. First we will prove that for any $k \in \mathbb{N}$ if the sequence $\{U_m\}_{m \geq 0}$ converges uniformly on $\mathcal{R}_{\varepsilon,n,\Lambda}^-(k)$ to some $U \in C^\omega(\mathcal{R}_{\varepsilon,n,\Lambda}^-(k))^2$, then U is the analytic extension of the solution \hat{U} found in Section 3.2.3.

Observe that for all $n \in \mathbb{N}$, both components of U_n are analytic on $\mathcal{R}_{\varepsilon,n,\Lambda}^-(k)$. Then the uniform limit, U , of any convergent subsequence has also components analytic on $\mathcal{R}_{\varepsilon,n,\Lambda}^-(k)$. But the same subsequence remains uniformly convergent even restricted on $\mathcal{R}_{\varepsilon,n,\Lambda}^-(r)$. Since on $\mathcal{R}_{\varepsilon,n,\Lambda}^-(r)$ its limit is \hat{U} , U has to be the analytic extension of \hat{U} . Evidently all convergent subsequences must converge to $\mathcal{R}_{\varepsilon,n,\Lambda}^-(r)$, so all convergent subsequences need to converge to the same limit.

Suppose that U_n does not converge to \hat{U} in $C^\omega(\mathcal{R}_{\varepsilon,n,\Lambda}^-)^2$. Then there exists $\kappa \in \mathbb{N}$, $\delta > 0$ and a subsequence $\{U_{m_k}\}_{k \geq 0}$ such that

$$\left\| U_{m_k} - \hat{U} \right\|_{\times 2, \mathcal{R}_{\varepsilon,n,\Lambda}^-(\kappa)} \geq \delta \quad (3.12)$$

for all $k \in \mathbb{N}$. But since the subsequence $\{U_{m_k}\}_{k \geq 0}$ is componentwise a family bounded in $\mathcal{R}_{\varepsilon,n,\Lambda}^-(\kappa)$, by virtue of Montel's theorem we can find a subsequence $\{U_{m_{k_l}}\}_{l \geq 0}$ in it, such that $\phi_{m_{k_l}} \rightarrow \hat{u}_x$ uniformly in $\mathcal{R}_{\varepsilon,n,\Lambda}^-(\kappa)$. Then by applying the theorem again to the second component of $\{U_{m_{k_l}}\}_{l \geq 0}$, we can get yet one subsequence $\{U_{m_{k_{l_i}}}\}_{i \geq 0}$ such that $\psi_{m_{k_{l_i}}} \rightarrow \hat{u}_y$ uniformly in $\mathcal{R}_{\varepsilon,n,\Lambda}^-(\kappa)$. But we saw that $U_{m_{k_{l_i}}} \rightarrow \hat{U}$ uniformly and this contradicts equation (3.12). Thus U_n has to converge to \hat{U} in $C^\omega(\mathcal{R}_{\varepsilon,n,\Lambda}^-)^2$. \square

From now on, both the solution \hat{U} and its analytic extension U will be denoted by \hat{U} .

3.2.6 Extension of the solution towards $+\infty$

To extend the solution to $+\infty$ we consider the inverse map (3.2). We see that this map is of the form of our original map. The crucial observation is that if $W_{\text{inv}}(t)$ is such that

$$W_{\text{inv}}(t+1) = F^{-1}(W_{\text{inv}}(t)),$$

from the form of F^{-1} we see that there exist v_x, v_y such that

$$W_{\text{inv}}(t) = \begin{pmatrix} \frac{v_x(t)}{t^2} \\ \frac{1}{b_0 t} + \frac{v_y(t)}{t^2} \end{pmatrix}.$$

We apply F to the equation and we shift the argument by one to get $W_{\text{inv}}(t-1) = F(W_{\text{inv}}(t))$. From there we get that $\tilde{W}_{\text{inv}}(t) = \tilde{W}(-t)$. Let $\tilde{W}(t) = \sum_{n \geq 0} c_n t^{-n-1}$. Then $\tilde{W}_{\text{inv}}(t) = \sum_{n \geq 0} (-1)^{n+1} c_n t^{-n-1}$ and from these we get that for the Borel transforms $\mathcal{B}[\tilde{W}]$ and $\mathcal{B}[\tilde{W}_{\text{inv}}]$ it holds $\mathcal{B}[\tilde{W}](s) = \mathcal{B}[\tilde{W}_{\text{inv}}](-s)$.

By the previous section we know that the Borel transform of the separatrices of F^{-1} converges in a neighbourhood of the origin and extends analytically on the Riemann surface $\mathcal{R}_{\varepsilon,n,\Lambda}^-$ with the same estimates. Because of the property above, we deduce that the Borel transform of the separatrices of F extends to a Riemann surface $\mathcal{R}_{\varepsilon,n,\Lambda}^+ = \{\gamma : R_I(\gamma) \in \mathcal{R}_{\varepsilon,n,\Lambda}^-\}$, with $R_I(\gamma)$ denoting the reflection of the

path γ across the imaginary axis.

3.2.7 The natural Riemann surface of the solution

We saw that the solution can be extended to both $\mathcal{R}_{\varepsilon,n,\Lambda}^-$ and $\mathcal{R}_{\varepsilon,n,\Lambda}^+$, which are subsets of $\mathcal{R}_{\varepsilon,n,\Lambda}$ with non empty intersection, so it means that it can be extended to $\mathcal{R}_{\varepsilon,n,\Lambda}$. Due to the previous results we have the following lemma.

Lemma 3.19. *Let $\Lambda \geq 1$, $n \in \mathbb{N}$ and $0 < \varepsilon < r$, with r being the radius of the disk on which we have contraction, as described in Section 3.2.3. Then $\{U_n\}_{n \geq 0} \rightarrow \hat{U}$ uniformly in all compact subsets of $\mathcal{R}_{\varepsilon,n,\Lambda}$.*

Given any path $\gamma \in \mathcal{R}_0$, we can find $\Lambda \geq 1$, $n \in \mathbb{N}$ and $0 < \varepsilon$ such that $\gamma \in \mathcal{R}_{\varepsilon,n,\Lambda}$. Moreover along any infinite paths γ that goes to infinity not vertically, U is of exponential growth.

So for $\hat{W} := \mathcal{B}[W]$, it holds

$$\hat{W}(s) = \begin{pmatrix} \hat{w}_x(s) \\ \hat{w}_y(s) \end{pmatrix} := \begin{pmatrix} s * \hat{u}_x(s) \\ \frac{1}{b_0} + s * \hat{u}_y(s) \end{pmatrix}.$$

Corollary 3.20. $\hat{W} \in (\widehat{RES}^{reg})^2$.

3.2.8 The Laplace transform of the solution

Through the Laplace transform we get 2 solutions of the equation (3.3), $\mathcal{L}^+[\hat{W}](t)$ and $\mathcal{L}^-[\hat{W}](t)$. We define them by

$$\begin{aligned} W^+(t) &= \mathcal{L}^+[\hat{W}](t) = \mathcal{L}^{(-\frac{\pi}{2}, \frac{\pi}{2})}[\hat{W}](t) = \int_0^{\infty e^{i\theta}} e^{-st} \hat{W}(s) dt, \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ W^-(t) &= \mathcal{L}^-[\hat{W}](t) = \mathcal{L}^{(\frac{\pi}{2}, \frac{3\pi}{2})}[\hat{W}](t) = \int_0^{\infty e^{i\theta}} e^{-st} \hat{W}(s) dt, \quad \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}). \end{aligned}$$

The domains of the solutions can be visualized by setting $\theta = \frac{\pi}{2}$ at Figure 2.3. Thus if we choose $t \in -i\mathbb{R}^+$, we can let $\theta \in (0, \frac{\pi}{2})$ for $\mathcal{L}^+[\hat{W}](t)$ and $\theta \in (\frac{\pi}{2}, \pi)$ for $\mathcal{L}^-[\hat{W}](t)$.

3.3 Singularities of the solution

3.3.1 The variational equation

An object central to this analysis is the fundamental solution of the variational of equation (3.3), namely a 2×2 matrix of determinant 1 that satisfies

$$\mathcal{V}(t+1) = F'(W(t)) \cdot \mathcal{V}(t). \quad (3.13)$$

This equation can be considered in all the 3 different representations we have. It can be viewed as an equation in the space of formal series, as an equation on the Borel plane and as an equation in the space of functions analytic in a sectorial neighbourhood of infinity. However in all 3 representations we already know a solution, namely $\dot{W}(t)$, so we can assume that $\mathcal{V}(t) = (\Xi(t), \dot{W}(t))$ and we need only to find $\Xi(t)$.

Solution in the space of formal series

Initially we look for a solution of the equation (3.13) in the space of formal series. The equation becomes

$$\tilde{\mathcal{V}}(t+1) = F'(\mathcal{W}(t)) \cdot \tilde{\mathcal{V}}(t)$$

and we have the following lemma.

Lemma 3.21. *There exists a unique $\tilde{\mathcal{V}} \in t^{-1}\mathbb{C}[[t^{-1}]]^2$ such that*

$$\tilde{\Xi}(t) = \begin{pmatrix} b_0 t^2 - 18 \frac{b_1}{b_0^2} + \frac{24b_1^2}{b_0^5} t^{-2} \\ -\frac{8a_0}{b_0^3} t^{-1} \end{pmatrix} + \frac{1}{t^2} \tilde{\mathcal{V}}(t).$$

satisfies both

$$\tilde{\Xi}(t+1) = F'(\mathcal{W}(t)) \cdot \tilde{\Xi}(t)$$

and

$$\det(\tilde{\Xi}(t), \dot{\mathcal{W}}(t)) = 1.$$

Proof. We write $\tilde{\Xi}(t) = (\tilde{\xi}_1(t), \tilde{\xi}_2(t))$, $\mathcal{W}(t) = (\tilde{\zeta}_1(t), \tilde{\zeta}_2(t))$ and $F'(\mathcal{W}(t)) = \tilde{D}(t)$. Then from $\det(\tilde{\Xi}(t), \mathcal{W}(t)) = 1$ we get

$$\tilde{\xi}_1(t+1) = \frac{\tilde{\zeta}_1(t)}{\tilde{\zeta}_2(t)} \tilde{\xi}_2(t) + \frac{1}{\tilde{\zeta}_2(t)}$$

and from $\tilde{\Xi}(t+1) = \tilde{D}(t) \cdot \tilde{\Xi}(t)$ combined with the above we get

$$\tilde{\xi}_2(t+1) = \left(\tilde{D}_{21}(t) \frac{\tilde{\zeta}_1(t)}{\tilde{\zeta}_2(t)} + \tilde{D}_{22}(t) \right) \tilde{\xi}_2(t) + \frac{\tilde{D}_{21}(t)}{\tilde{\zeta}_2(t)}.$$

Evidently $\tilde{\zeta}_2$ satisfies the homogeneous part of the above finite difference equation. So we set $\tilde{\xi}_2(t) = c(t) \tilde{\zeta}_2(t)$ and by substituting this in the equation we get

$$c(t+1) - c(t) = \frac{\tilde{D}_{21}(t)}{\tilde{\zeta}_2(t+1) \tilde{\zeta}_2(t)}.$$

This equation defines c up to the addition of a constant. It can be checked³ that the right hand side of this equation is $-8a_0/b_0^2 + O(t^{-2})$ and from this it holds that

$$c(t) = \frac{8a_0}{b_0^2} t + O(t^{-1}) \in t^1 \mathbb{C}[[t^{-1}]]$$

is the unique solution without constant term. This determines $\tilde{\xi}_2$, which then determines $\tilde{\xi}_1$. \square

Solution on the Borel plane

To find the Borel transform of the formal series solution we need to take the Borel transform of (3.13)

$$e^{-s} \check{\mathcal{V}}(s) = F'_*[\hat{W}] * \check{\mathcal{V}}(s) \quad (3.14)$$

with $F'_*[\hat{W}]$ the Borel transform of $F'(\mathcal{W}(t))$. Denote by $\check{\Xi}$ the Borel transform of $\tilde{\Xi}$ and we get the following lemma.

Lemma 3.22. *For all $n \in \mathbb{N}$, $\Lambda \geq 1$ and $1 > \varepsilon > 0$, there exists $\hat{\mathcal{V}} \in (\widehat{RES}^{reg})^2$*

³This is the reason that we need F to agree with the normal form up to degree 4.

with variation that is of exponential type $\tau(\Lambda, \varepsilon)$ on $\mathcal{R}_{\varepsilon, n, \Lambda}$ such that

$$\check{\Xi}(s) = \begin{pmatrix} b_0 \delta^{(2)}(s) - 18 \frac{b_1}{b_0^2} \delta(s) + \frac{24b_1^2}{b_0^5} \flat s \\ -\frac{8a_0}{b_0^3} \flat 1 \end{pmatrix} + \flat \left(s * \hat{\mathcal{V}}(s) \right) \quad (3.15)$$

and

$$\check{\mathcal{V}}(s) = \left(\check{\Xi}(s), -s\hat{W}(s) \right).$$

Proof. The proof uses the same machinery as in the previous section, but since the equation is linear, the whole process is simpler. Here we will sketch the proof and highlight the differences.

We already know that $-s\hat{W}(s)$ satisfies the equation (3.14) since it is the derivative of $\hat{W}(s)$, so we just need to check $\check{\mathcal{V}}$.

First note that it was proven in 3.2 that $F_*[\hat{W}]$ is of exponential type and that this depends on the radius of analyticity of F , on b_0 , on ε and Λ . Using the same bounds we see that $F'_*[\hat{W}]$ is of the same exponential type.

We write

$$F'(x, y) = G(x, y) + R(x, y)$$

with $R(x, y)$ a 2×2 matrix whose components have Taylor series around the origin that begin with order 4.

Let $G_*[\hat{W}]$ and $R_*[\hat{W}]$ be the Borel transforms of $G(\mathcal{W}(t))$ and $R(\mathcal{W}(t))$ respectively, Then we have $F'_*[\hat{W}] = G_*[\hat{W}] + R_*[\hat{W}]$ and there exists $r > 0$ such that $R_*[\hat{W}] \in \mathcal{B}_3(\mathbb{D}_r)$. We substitute (3.15) in (3.14) to get after canceling

$$L[\hat{\mathcal{V}}](s) = A(s) + R_*[\hat{W}] * P(s) + B * \hat{\mathcal{V}}(s) + R_*[\hat{W}] * \hat{\mathcal{V}}(s) \quad (3.16)$$

with $A, B \in (\mathcal{B}_1(\mathbb{D}_r))^2$ and L the linear operator defined in 3.7.

The operator L^{-1} maps $\mathcal{B}_{n+1}(\mathbb{D}_r)$ to $\mathcal{B}_n(\mathbb{D}_r)$, so it can be applied on (3.16) and then it becomes a map from $\mathcal{B}_0(\mathbb{D}_r)$ to itself.⁴ Then by bounding the operator and choosing a small enough r it can be turned into a contraction.

For the extension beyond \mathbb{D}_r we just need to bound $\hat{\mathcal{V}}$ by $Me^{a|s|}$ with big enough M and $a > \alpha/b_0$ and use the same procedure as above. This way we get the extension

⁴Note that $R_*[\hat{W}] * \hat{\mathcal{V}} \in (\mathcal{B}_1(\mathbb{D}_r))^2$.

to $-\infty$. To get the extension to $+\infty$ we apply the result to the inverse map. \square

Solution in the space of functions analytic in a sectorial neighbourhood of infinity

Using the previous lemma we get the existence of two functions analytic in the same domains as the solutions of the separatrix equation and both admit $\tilde{\mathcal{V}}(t)$ as asymptotic. One corresponds to the fundamental solution of the variational equation around W^+ and the other one corresponds to the fundamental solution of the variational equation around W^- . In their respective domains they satisfy (3.13). The difference of these two solutions, where they are both defined, is exponentially small.

3.3.2 Non-homogeneous variational equation

We call non-homogeneous variational equation, an equation of the form

$$X(t+1) = F'(\mathcal{W}(t)) \cdot X(t) + B(t),$$

with $B \in (\widehat{\text{RES}}^{\text{sim}})^2$ or $B \in (\widehat{\text{RES}}^{\text{sim}})^{2 \times 2}$, with $\text{var} \hat{B}$ of exponential type $\tau(\Lambda, \varepsilon)$ on $\mathcal{R}_{\varepsilon, n, \Lambda}$. To solve this type of equation we write $X(t) = \tilde{\mathcal{V}}(t) \cdot Y(t)$ and we get

$$\begin{aligned} X(t+1) &= \tilde{\mathcal{V}}(t+1) \cdot Y(t+1) \\ &= F'(\mathcal{W}(t)) \cdot X(t) + B(t) \\ &= F'(\mathcal{W}(t)) \cdot \tilde{\mathcal{V}}(t) \cdot Y(t) + B(t) \\ &= \tilde{\mathcal{V}}(t+1) \cdot Y(t) + B(t) \end{aligned}$$

and from this we get

$$Y(t+1) = Y(t) + \tilde{\mathcal{V}}^{-1}(t+1) \cdot B(t).$$

Note that since the determinant of $\tilde{\mathcal{V}}$ is 1 its inverse is

$$\tilde{\mathcal{V}}^{-1}(t) = \begin{pmatrix} \dot{\mathcal{W}}_y(t) & -\dot{\mathcal{W}}_x(t) \\ -\mathcal{V}_y(t) & \mathcal{V}_x(t) \end{pmatrix}.$$

Of course we can view the same equations on the Borel plane. They become

$$e^{-s}\check{X}(s) = F'[\check{W}] * \check{X}(s) + \check{B}(s),$$

$\check{X}(s) = \check{\mathcal{V}} * \check{Y}(s)$ and after substitution we get

$$e^{-s}\check{Y}(s) = \check{Y}(s) + (e^{-s}\check{\mathcal{V}}^{-1}) * \check{B}(s).$$

This can be solved immediately and we get

$$\check{Y}(s) = \frac{e^s}{1-e^s} \left((e^{-s}\check{\mathcal{V}}^{-1}) * \check{B}(s) \right),$$

so finally

$$\check{X}(s) = \check{\mathcal{V}} * \left(\frac{e^s}{1-e^s} \left((e^{-s}\check{\mathcal{V}}^{-1}) * \check{B}(s) \right) \right)(s).$$

This implies that $\check{X} \in (\widehat{\text{RES}})^2$ and that $\text{var}\check{X}$ is of exponential type $\tau(\Lambda, \varepsilon)$ on $\mathcal{R}_{\varepsilon, n, \Lambda}$. However we do not get directly that $X \in (\widehat{\text{RES}}^{\text{sim}})^2$. For this to hold, the variance of $(e^{-s}\check{\mathcal{V}}^{-1}) * \check{B}(s)$ has to have a simple root at the origin.

Remark. When we try to solve the non-homogeneous variational equation, we multiply by $\frac{e^s}{1-e^s}$. If we restrict our view only in SING, this is the same as multiplying by $\frac{1}{s}$. This operation does not alter the type of terms of the form $\delta^{(n)}$. It also does not alter the type of terms of the form $\text{sing}_0(s^n \log(s))$ with $n \geq 1$. The problem arises with the term $\text{sing}_0(\frac{\log(s)}{2\pi i})$, since then we get $\text{sing}_0(\frac{1}{s} \frac{\log(s)}{2\pi i})$. If we look at the same operation on formal model we see that it corresponds to integrating $\frac{1}{t}$. This creates a logarithmic term so the formal solution is no longer in $\mathbb{C}[t][[t^{-1}]]^2$. Which leads to a formal solution being a series of powers of t and powers of logarithm of t .

3.3.3 The first singularity of \hat{W}

We know that $\hat{W} \in (\widehat{\text{RES}}^{\text{reg}})^2$ so we can use the alien derivations. We are interested in the singularity at ω_1 . Since \hat{W} satisfies $e^{-s}\hat{W}(s) = F_*[\hat{W}](s)$, its alien derivative $\Delta_{\omega_1}[\hat{W}]$ satisfies $e^{-s}\Delta_{\omega_1}[\hat{W}](s) = F_*'[\hat{W}] * \Delta_{\omega_1}[\hat{W}](s)$. Similarly to Example 2.22 we get the existence of $C \in \mathbb{C}^2$, $C = (\theta, \kappa)$ such that

$$\Delta_{\omega_1}[\mathcal{W}](t) = \tilde{\mathcal{V}}(t) \cdot C.$$

From this we get $\Delta_{\omega_1}[\hat{W}](s) = \theta \check{\mathcal{V}}(s) - \kappa s \hat{W}(s)$.

Analyticity of the Stokes constant with respect to parameters

As we saw the first singularity of the solution is determined by two constants. It is a natural question to ask how these constants depend on the parameter on an one parameter family of maps at 1:3 resonance.

To this end we define the norm

$$\|f\|_{n,m} := \sup_{s \in \mathbb{D}_{2\pi}} |s^{-n}(s^2 + 4\pi^2)^m f(s)|$$

and we define $\mathfrak{B}_{n,m}$ to be the space of all functions with bounded n, m -norm in $\mathbb{D}_{2\pi}$.

By recalling the definitions of \mathcal{F} and L^{-1} in sections 3.2.1 and 3.2.2 and using the same bounding techniques, we can see that \mathcal{F} is a bounded map from $\mathfrak{B}_{0,5}$ to $\mathfrak{B}_{2,3}$ and L^{-1} is a bounded linear map from $\mathfrak{B}_{2,3}$ to $\mathfrak{B}_{0,5}$.

Then the map⁵ \mathcal{F} is a bounded map from $\mathfrak{B}_{0,5}$ to itself. This implies that the limit of the sequence $\{U_n\}_{n \geq 0}$ defined in equation (3.10) is a fixed point of the map \mathcal{F} . Using the same arguments we can show that the sequence converges uniformly with the norm $\|\cdot\|_{0,5}$. Then we define⁶

$$\theta_n = \lim_{\sigma \nearrow 2\pi} (s - 2\pi i)^5 \phi_n(i\sigma).$$

Recall that ϕ_n is the first component of U_n . Trivially we see that $\lim_{n \rightarrow \infty} \theta_n = \theta$.

Now we assume that instead of a single map F we have an analytic family of maps F_λ such that for all appropriate λ there exists an area-preserving map f_λ at 1:3 resonance such that $F_\lambda = f_\lambda^3$. Moreover we assume that all the maps of the family are analytic in a complex neighbourhood of the origin in \mathbb{C}^2 and uniformly bounded.

This implies that now θ_n 's are a function of λ and $\lim_{n \rightarrow \infty} \theta_n(\lambda) = \theta(\lambda)$. Now we notice that in all bounds we use the radius of convergence of F_λ and the supremum of F_μ and not λ . This implies that since all maps in the family have a non-zero lower bound for their radii of convergence, the sequence $\{\theta_n\}_{n \geq 0}$ converges uniformly to θ . This of course implies that θ is an analytic function of λ . This reasoning can be extended to any finite number of parameters.

Remark. The next obvious step is to try to extend this result to a family with “infinitely many parameters”. This extension however is non-trivial since in infinite

⁵ Recall that $\mathcal{F} = L^{-1} \circ (\mathcal{A} + \mathcal{F})$.

⁶ The notation $\sigma \nearrow 2\pi$ implies that σ is real and always less than 2π .

dimensional spaces there are multiple non-equivalent topologies that can be defined. This means that in order to extend this result, the topology in the “parameter” space has to be carefully chosen. In the current text we make no claim in this direction.

First singularity of the fundamental solution

As we saw above, to get the first singularity of a solution of the non-homogeneous variational equation we need to know the first singularity of the fundamental solution.

To get this we just apply the alien derivative Δ_{ω_1} to (3.14). We get

$$e^{-s} \Delta_{\omega_1}[\check{\mathcal{V}}](s) = F'_*[\hat{W}] * \Delta_{\omega_1}[\check{\mathcal{V}}](s) + \Delta_{\omega_1}[F'_*[\hat{W}]] * \check{\mathcal{V}}(s)$$

and using the results of the previous section for the non-homogeneous variational equations we get

$$\Delta_{\omega_1}[\check{\mathcal{V}}](s) = \check{\mathcal{V}} * \left(\frac{e^s}{1 - e^s} \left((e^{-s} \check{\mathcal{V}}^{-1}) * \Delta_{\omega_1}[F'_*[\hat{W}]] * \check{\mathcal{V}} \right) \right) (s).$$

Up to now we saw only simple singularities and this information came from the formal expansion. To get the same information on the Borel plane is much more difficult and we will see why.

To check whether $\Delta_{\omega_1}[\check{\mathcal{V}}] \in (\widehat{\text{RES}}^{\text{sim}})^{2 \times 2}$ we can check whether the variance of $(e^{-s} \check{\mathcal{V}}^{-1}) * \Delta_{\omega_1}[F'_*[\hat{W}]] * \check{\mathcal{V}}(s)$ has a simple root at the origin or whether $\check{\mathcal{V}}^{-1}(t + 1) \cdot \Delta_{\omega_1}[F'(\mathcal{W})](t) \cdot \check{\mathcal{V}}(t)$ contains the term $\frac{1}{t}$. It turns out that it can be shown that $\Delta_{\omega_1}[\check{\mathcal{V}}] \in (\widehat{\text{RES}}^{\text{sim}})^{2 \times 2}$ and $\Delta_{\omega_1}[\check{\mathcal{V}}](t) \in (t^6 \mathbb{C}[[t^{-1}]])^{2 \times 2}$ if the map F agrees with the normal form at least up to order 8.

Remark. This hints to the situation where the more powers the map F agree with the normal form, the more singularities are found to be simple. Thus the question about the singularities of the map we assumed in the beginning of this chapter, a map that agrees with the normal form up to degree 3 is raised, Let us denote, just for this remark, by F_n a map that agrees with the normal form up to degree n . If we look at the relation between F_3 and F_8 we will see that there is an analytic transformation⁷ that changes F_8 to F_3 . Then this transformation induces a trans-

⁷ The transformation to put the map to the normal form up to a finite order can be constructed as the time 1 flow of a polynomial Hamiltonian. Of course if we try to push the order to infinity we will get a formal transformation as a result.

formation between the fundamental solution of the variational equation for F_8 and the fundamental solution of the variational equation for F_3 and this transformation is also analytic. By proceeding this way we will see that there is an analytic transformation between the first singularities of these fundamental solutions. By looking at the formal model again we see that the asymptotics for the first singularities of the fundamental solutions are conjugated by a power series and conjugacy by power series cannot create logarithmic terms. So we conclude that $\Delta_{\omega_1}[\check{\mathcal{V}}]$ is simple.

We should note here that the goal of this analysis is to get an asymptotic for the difference of the separatrices of the map and this does not require precise knowledge of all the singularities of the solution. It suffices to know Δ_{ω_1} precisely and know the biggest term for $\Delta_{\omega_n}^+$ with $n > 1$ and this is how we will proceed.

3.3.4 Further singularities of \hat{W}

The singularities that interest us are $\Delta_{\omega_n}^+ U$ for $n > 1$. We know that $\{U_j\}_{j \geq 0}$ converges to the solution U , so we can look at the limit of the sequence $\{\Delta_{\omega_n}^+ U_j\}_{j \geq 0}$. By definition we have

$$U_{n+1} = L^{-1} [\mathcal{A} + \mathcal{F}[U_n]]. \quad (3.17)$$

Lemma 2.20 tells us how the operator $\Delta_{\omega_n}^+$ acts on convolution, so we need to see how it acts after the operator L^{-1} . The operator L^{-1} constructed by two operators, \mathcal{K} and \mathcal{I} , seen below,

$$\begin{aligned} \mathcal{K}[X](\gamma) &= \frac{e^{\pi(\gamma)}}{1 - e^{\pi(\gamma)}} X(\gamma), \\ \mathcal{I}[X](\gamma) &= \frac{4e^{\pi(\gamma)}}{(1 - e^{\pi(\gamma)})^5} \int_0^1 e^{\pi(\gamma^t)} (1 - e^{\pi(\gamma^t)})^3 X(\gamma^t) \dot{\gamma}(t) dt. \end{aligned}$$

Suppose that both X and Y are entire functions, then $\mathcal{K}[X]$ and $\mathcal{I}[Y]$ are meromorphic functions for which it holds

$$\begin{aligned} \Delta_{\omega_n}^+ [\mathcal{K}[X]] &= O(\delta), \\ \Delta_{\omega_n}^+ [\mathcal{I}[Y]] &= O(\delta^{(5)}). \end{aligned}$$

This notation should be understood as denoting the fastest growing term as s tends to ω_n .

Suppose now that $\Delta_{\omega_n}^+ [\mathcal{K}[X]] = O(\delta^{(k)})$ and $\Delta_{\omega_n}^+ [\mathcal{I}[Y]] = O(\delta^{(m)})$ with $k > 1$ and

$m > 5$. Then we have

$$\Delta_{\omega_n}^+[\mathcal{K}[X]] = O(\delta^{(k+1)}).$$

Notice that the definition of \mathcal{I} involves an integral of X multiplied by an entire function, Moreover, this function has a triple root at any ω_n . So we see that the integral drops the order of the pole by 4 and then the division by $(1 - e^s)^5$ raises it by 5, so in the end we have

$$\Delta_{\omega_n}^+[\mathcal{I}[Y]] = O(\delta^{(m+1)}).$$

For the first one we have the straightforward relation

$$\Delta_{\omega_n}^+[\mathcal{K}[X]](s) = \Delta_{\omega_n}^+ \left[\frac{e^{\pi(\gamma)}}{1 - e^{\pi(\bar{\gamma})}} \right] \Delta_{\omega_n}^+[X](s),$$

which can be simplified as

$$\Delta_{\omega_n}^+[\mathcal{K}[X]](s) = \frac{1}{s} \Delta_{\omega_n}^+[X](s).$$

For the operator \mathcal{I} , let us first assume that X is an entire function. Then clearly $[\mathcal{I}[X]]$ is a meromorphic function and $\Delta_{\omega_n}^+[\mathcal{I}[X]] = O(\delta^{(5)})$. Let us now assume that $X \in \widehat{\text{RES}}$ and that $\Delta_{\omega_n}^+[X] = O(\delta^{(m)})$ with $m \geq 5$. First we notice that all other singularities play no role since the definition of \mathcal{I} involves an integral of X multiplied by an entire function, Moreover, this function has a triple root at any ω_n . So we see that the integral drops the order of the pole by 4 and then the division by $(1 - e^s)^5$ raises it by 5, so in the end we have $\Delta_{\omega_n}^+[\mathcal{I}[X]] = O(\delta^{(m+1)})$.

We also need to notice that if $X = O(\delta^{(m)})$ and $Y = O(\delta^{(k)})$, with $m, k \in \mathbb{N}$, then $X * Y = O(\delta^{m+k-1})$. As a first attempt we look at the sequence $\{\Delta_{\omega_2}^+ U_j\}_{j \geq 0}$. We have

$$\begin{aligned} \mathcal{F} \left[\begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} \right] &= \begin{pmatrix} (6s + O(s^2)) * \phi_n - 2b_0(s + O(s^2)) * \phi_n * \psi_n \\ (2s^2 + O(s^3)) * \psi_n - b_0(s + O(s^2)) * (\phi_n^{*2} - \psi_n^{*2}) \end{pmatrix} \\ &\quad + \begin{pmatrix} + \frac{6b_0^2}{3!}(s^3 + O(s^4)) * (\phi_n * \psi_n^{*2} + \phi_n^{*3}) + \dots \\ + \frac{6b_0^2}{3!}(s^3 + O(s^4)) * (\phi_n^{*2} * \psi_n + \psi_n^{*3}) + \dots \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}\Delta_{\omega_2}^+[s * \phi_n] &= s * \Delta_{\omega_2}^+[\phi_n], \\ \Delta_{\omega_2}^+[s * \phi_n^{*2}] &= 2s * \phi_n * \Delta_{\omega_2}^+[\phi_n] + s * \Delta_{\omega_1}^+[\phi_n]^{*2}, \\ \Delta_{\omega_2}^+[s^3 * \phi_n^{*3}] &= 3s^3 * \phi_n^{*2} * \Delta_{\omega_2}^+[\phi_n] + 3s^3 * \phi_n * \Delta_{\omega_1}^+[\phi_n]^{*2}.\end{aligned}$$

Then we apply $\Delta_{\omega_2}^+$ on equation (3.17) starting with $U_0 = 0$ so we find that for $m > 2$ we have

$$\Delta_{\omega_2}^+ U_m = \begin{pmatrix} O(\delta^{(5)}) \\ O(\delta^{(8)}) \end{pmatrix}.$$

Then we can repeat the process for $n > 2$ and by induction we find that

$$\Delta_{\omega_n}^+ U = \begin{pmatrix} O(\delta^{(k_x(n))}) \\ O(\delta^{(k_y(n))}) \end{pmatrix},$$

with $k_x(n) = \frac{1}{2}(6n - 3(-1)^n + 1)$ and $k_y(n) = \frac{1}{2}(6n + 3(-1)^n + 1)$. So for \hat{W} we get

$$\Delta_{\omega_n}^+ \hat{W} = \begin{pmatrix} O(\delta^{(l_x(n))}) \\ O(\delta^{(l_y(n))}) \end{pmatrix},$$

with $l_x(n) = \frac{1}{2}(6n - 3(-1)^n - 1)$ and $l_y(n) = \frac{1}{2}(6n + 3(-1)^n - 1)$.

What is actually of importance here is that the order of the poles grow linearly with n . In practice any sub-exponential growth would have been sufficient.

3.4 Splitting of the separatrices

Let $\delta(t) = \mathcal{L}^+[\hat{W}](t) - \mathcal{L}^-[\hat{W}](t)$. Choose $t \in -i\mathbb{R}^+$ big enough and $\theta \in (0, \frac{\pi}{2})$ such that

$$\delta(t) = \int_0^{\infty e^{i\theta}} e^{-st} \hat{W}(s) dt - \int_0^{\infty e^{i(\pi-\theta)}} e^{-st} \hat{W}(s) dt$$

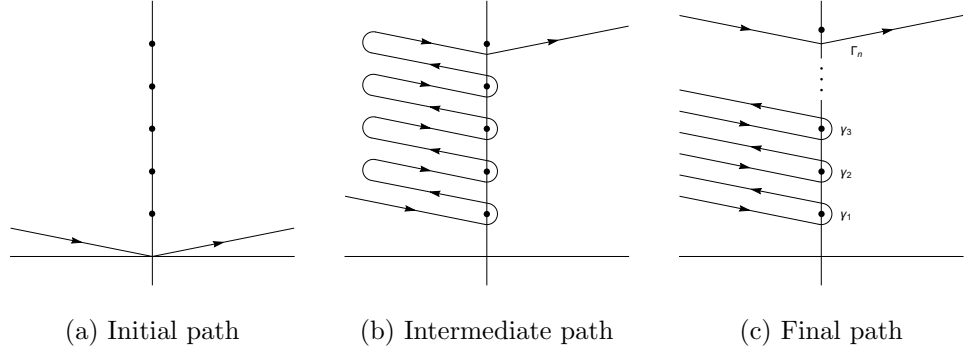


Figure 3.1: Deformation of the integration path

$$\begin{aligned}
&= \int_0^{\infty e^{i\theta}} e^{-st} \hat{W}(s) dt + \int_{\infty e^{i(\pi-\theta)}}^0 e^{-st} \hat{W}(s) dt \\
&= \int_{\Gamma_0} e^{-st} \hat{W}(s) dt
\end{aligned}$$

Then we deform the path of integration Γ as shown in Figure 3.1. From this deformation we get

$$\delta(t) = \sum_{j=1}^n \int_{\gamma_j} e^{-st} \hat{W}(s) dt + \int_{\Gamma_n} e^{-st} \hat{W}(s) dt.$$

By pushing n to infinity we get

$$\delta(t) \asymp \sum_{j \geq 1} \int_{\gamma_j} e^{-st} \hat{W}(s) dt.$$

Convergence is not guaranteed for the infinite sum but the integral over Γ_m can be bounded by $e^{-\omega_{m+1} t}$. We know that $\int_{\gamma_m} e^{-st} \hat{W}(s) dt = e^{-\omega_m t} \mathcal{L}^{\pi-\theta}[\Delta_{\omega_m}^+[\hat{W}]](t) \asymp e^{-\omega_m t} \Delta_{\omega_m}^+[\mathcal{W}](t)$. Since $e^{-\omega_m t}$ decreases exponentially and $\Delta_{\omega_m}^+[\mathcal{W}](t)$ increases linearly, only the first term is needed for the asymptotic so we have

$$\delta(t) \asymp e^{-\omega_1 t} \Delta_{\omega_1}[\mathcal{W}](t) + O(e^{-\omega_2 t} \Delta_{\omega_2}[\mathcal{W}](t)).$$

From this we get that

$$\theta = \lim_{t \rightarrow \infty} e^{2\pi t} \omega(W^+(-it) - W^-(-it), \dot{W}^-(-it)).$$

Remark. This last relation comes from the fact that

$$\theta = e^{2\pi t} \omega(W^+(-it) - W^-(-it), \dot{W}^-(-it)) + O(t^5 e^{-2\pi t}).$$

This holds because the extra term of $\delta(t)$, namely $O(e^{-\omega_2 it} \Delta_{\omega_2}[\mathcal{W}](it))$, is of order $O(t^7 e^{-4\pi t})$. This gets multiplied with $\dot{W}^-(-it)$, which is of order $O(t^{-2})$, and then multiplied by $O(e^{2\pi t})$.

Chapter 4

Splitting of separatrices of an area-preserving map close to 3:1 resonance

4.1 Setup

Let f_μ be an analytic family of area-preserving maps and that f_0 is an area-preserving map at 1:3 resonance. We fix $M \in \mathbb{N}$ and define $F_\mu = f_\mu^3$. Then F_0 is a tangent to identity area-preserving map. We assume that it agrees with the normal form in Proposition 1.5 up to degree $N = 4M + 35$ and that the Stokes constant of the resonant map F_0 does not vanish.

the map F_μ is analytic in x , y and μ , so F_μ can be decomposed in two ways, namely $F_\mu(x, y) = \sum_{n \geq 0} \mu^n F_n(x, y)$ and $F_\mu(x, y) = \sum_{n \geq 1} \mathcal{F}_n(\mu; x, y)$. Here F_n are real analytic functions¹ independent of ε and \mathcal{F}_n are polynomials of degree n homogeneous in its three variables.

4.2 Notation

In the analysis a handful of objects appear and it is useful to give a list of them, fixing the notation. No proper definitions or proofs will be provided here, these will

¹ There is an abusion of notation here since the subscript of F can denote either a real or a natural number. However it will be clear by the context which case is considered and for the case $\varepsilon = 0$ both notations agree.

appear later in the chapter.

The starting point is the family of area preserving maps F_μ . We assume that it is non-degenerate, i.e. that it can be transformed to agree with the normal form given in Proposition 1.5 up to an arbitrary order by an analytic transformation.

Choosing a small $\mu \neq 0$ we get three saddle points with $\lambda_\mu > 1$ being their largest eigenvalue. We set $\varepsilon = \log(\lambda_\mu)$ and by the implicit function theorem we can write the parameter μ as a function of ε . Throughout the text we assume that the parametrization of F_μ is changed from μ to ε which is in a sense the natural parametrization. We will also assume that $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. We treat ε_0 as if it was fixed but we are allowed to decrease it if the need arises. We also need to choose a technical constant² $\Lambda > 1$ such that $\Lambda^2 \varepsilon_0 < 1$. We are also allowed to increase Λ if the need arises making sure that ε_0 will be decreased proportionately.

It should be noted that the way this parametrization is analytic but we do not intent to actually cross 0. This does not restrict the application of the result because if F_μ is a family of maps as described above then $F_{-\mu}$ is also such a family. So we conduct our analysis for $\mu \in (0, \mu_0)$ or $\mu \in (-\mu_0, 0)$ for some $\mu_0 > 0$.

Given F_ε , we can construct $\tilde{H}(\varepsilon; x, y)$ which is the formal Hamiltonian of the normal form. From this we can construct its formal time-1 flow \tilde{F}_ε .

The central objects in this analysis are the functions $W^-(\varepsilon; \tau)$ and $W^+(\varepsilon; \tau)$ which correspond to the vertical heteroclinic connection of in the normal form. They satisfy the equation $W^\pm(\varepsilon; \tau + 1) = F_\varepsilon(W^\pm(\varepsilon; \tau))$. Unless it is explicitly stated, it will be assumed from now on that the separatrices are parametrized with step 1 as above. We fix the parametrization by asking that $W^+(\varepsilon; 0)$ is the point where the stable separatrix meets the horizontal axis for the first time. Similarly $W^-(\varepsilon; 0)$ is the point where the unstable separatrix meets the horizontal axis for the first time.

There are four formal solutions considered: \tilde{W} , $\tilde{\mathcal{W}}$, $\tilde{\mathfrak{W}}$ and $\tilde{\mathbb{W}}$. The first, \tilde{W} , satisfies $\tilde{W}(\varepsilon; \tau + 1) = F_\varepsilon(\tilde{W}(\varepsilon; \tau))$ and the second, $\tilde{\mathcal{W}}$, satisfies $\tilde{\mathcal{W}}(\varepsilon; \tau + 1) = \tilde{F}_\varepsilon(\tilde{\mathcal{W}}(\varepsilon; \tau))$. Both of those are formal series in $\tanh(\frac{\varepsilon\tau}{2})$ and ε , so both have a singularity at πi . For the third one we change the parametrization from τ to t with $t = \tau - \frac{\pi i}{\varepsilon}$, then $\tilde{\mathfrak{W}}$ is just \tilde{W} with $\tanh(\frac{\varepsilon\tau}{2})$ expanded as Laurent series close to the singularity. Then $\tilde{\mathfrak{W}}$ satisfies $\tilde{\mathfrak{W}}(\varepsilon; t + 1) = F_\varepsilon(\tilde{\mathfrak{W}}(\varepsilon; t))$. Finally $\tilde{\mathbb{W}}$ is $\tilde{\mathcal{W}}$ with $\tanh(\frac{\varepsilon\tau}{2})$ expanded as Laurent series close to the singularity. Notice that the first component of $\tilde{\mathcal{W}}$ and $\tilde{\mathbb{W}}$ is even in τ and t respectively and the second component is odd.

² The role of this constant is to fine tune our domain, so we can have the appropriate bounds for our approximations.

There are also two linear equations that play an important role to the proof. These are

$$U(\varepsilon; \tau + 1) = A(\varepsilon; \tau) \cdot U(\varepsilon; \tau), \quad (4.1)$$

$$V(\varepsilon; \tau + 1) = D(\varepsilon; \tau) \cdot V(\varepsilon; \tau), \quad (4.2)$$

with

$$A(\varepsilon; \tau) = \int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) ds$$

and

$$D(\varepsilon; \tau) = F'_\varepsilon(W^-(\varepsilon; \tau)).$$

Evidently $\delta = W^+ - W^-$ satisfies the first one and \dot{W}^- satisfies the second. We denote by U the fundamental solution of the first and by V the fundamental solution of the second, normalized by $\det U = \det V = 1$. Moreover we have $U \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta$ and $V \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \dot{W}^-$.

We will see that there exists an open domain in variable τ that contains the origin and goes ε close to the singularities $\pm \frac{\pi}{\varepsilon} i$ in which both U and V are analytic and their difference is small.

If we look at equations (4.1) and (4.2) in the formal setting we see that the formal matrices \tilde{A} and \tilde{D} coincide, so we look at the formal equation close to the singularity

$$\tilde{W}(\varepsilon; t + 1) = \tilde{D}(\varepsilon; t) \cdot \tilde{W}(\varepsilon; t).$$

We denote by \tilde{W} its fundamental solution that satisfies $\det \tilde{W} = 1$ and $\tilde{W} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \dot{\tilde{W}}$.

4.3 Main result and outline of the proof

The main result of this chapter is the asymptotic behaviour of the Lazutkin homoclinic invariant, see section 1.2, as ε goes to 0. The following theorem summarizes the result.

Theorem 4.1. *Let F_ε be an analytic family of area-preserving maps as described above and let Ω be the Lazutkin homoclinic invariant of the map. Then there exist $\varepsilon_0 > 0$ and real constants ϑ_n such that for any $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$\Omega(\varepsilon) = \left(\sum_{n=0}^M \vartheta_n \varepsilon^n + O(\varepsilon^{M+1}) \right) e^{-\frac{2\pi^2}{\varepsilon}}.$$

Moreover $\vartheta_0 = 4\pi|\theta|$, where θ is the Stokes constant of the resonant map.

This theorem leads to the following direct corollary.

Theorem 4.2. *Let g_μ be an analytic family of area-preserving maps that agrees with the normal form up to degree 4 and that g_0 is a non-degenerate area-preserving map at resonance 1:3. We set $G_\mu = g_\mu^3$. For $\mu \neq 0$ let λ_μ denote the largest eigenvalue of its saddle points and let Ω be the Lazutkin homoclinic invariant of the map. If the Stokes constant θ of the resonant map does not vanish, then there exist $\mu_0 > 0$ and real constants ϑ_n such that for any $\mu \in (-\mu_0, \mu_0) \setminus \{0\}$ and any $M \in \mathbb{N}$ it holds*

$$\Omega(\mu) = \left(\sum_{n=0}^M \vartheta_n (\log \lambda_\mu)^n + O((\log \lambda_\mu)^{M+1}) \right) e^{-\frac{2\pi^2}{\log \lambda_\mu}}.$$

Moreover $\vartheta_0 = 4\pi|\theta|$.

Proof. For any $M \in \mathbb{N}$ there exists an analytic symplectic transformation that changes G_μ to F_μ with the properties assumed by Theorem 4.2. Then since the Ω is an invariant the result translates directly to G_μ . \square

The proof of Theorem 4.2 consists roughly of the following steps. First we prove the existence of the formal solutions described above for the separatrix equation. Then we prove that the formal solution close to the singularity approximates the separatrix close to the singularity. The process to do so is called complex matching, see [GL01].

Then we introduce the function

$$\Theta^-(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau)).$$

We will see that this function is approximately periodic and the derivative of this function at a homoclinic point gives the homoclinic invariant. This enables us to compute its “first Fourier coefficient”. We calculate the value of this function close to the singularity where it is polynomially small with ε and finally we translate the result at the real axis where we see it is exponentially small.

4.4 Formal solution of the separatrix equation

By Proposition 1.5 we know that there is a formal change of coordinates that transforms the map F_μ to the 1-flow of a formal Hamiltonian $\tilde{H}(\varepsilon; x, y)$. By solving

formally Hamilton's equations we have the following proposition.

Lemma 4.3. *Let $\sigma = \tanh(\frac{\varepsilon\tau}{2})$ and let $\tilde{H}(\mu(\varepsilon); x, y)$ be a formal Hamiltonian as described in Proposition 1.5. Then there exist a real formal power series such that $\mu(\varepsilon) = \sum_{n \geq 1} \mu_n \varepsilon^n$ and a real formal solution $\tilde{\mathcal{W}}(\varepsilon; \tau) = (\tilde{x}(\varepsilon; \tau), \tilde{y}(\varepsilon; \tau))$ of Hamilton's equations³*

$$\begin{aligned}\dot{x} &= \partial_y \tilde{H}(\mu(\varepsilon); x, y), \\ \dot{y} &= -\partial_x \tilde{H}(\mu(\varepsilon); x, y),\end{aligned}$$

such that

$$\begin{aligned}\tilde{x}(\varepsilon; \tau) &= \sum_{n \geq 1} \varepsilon^n P_n(\sigma), \\ \tilde{y}(\varepsilon; \tau) &= \sum_{n \geq 1} \varepsilon^n Q_n(\sigma),\end{aligned}$$

with $P_n(\sigma)$ even polynomials of degree $2\lfloor \frac{n}{2} \rfloor$, $Q_n(\sigma)$ odd polynomials of degree $2\lfloor \frac{n+1}{2} \rfloor - 1$ and $P_1(\sigma) = \frac{1}{2\sqrt{3}b_{0,0}}$, $Q_1(\sigma) = \frac{\sigma}{2b_{0,0}}$, $\mu_1 = \frac{1}{2\sqrt{3}a_{0,1}}$. Moreover P_n , Q_n and μ_n depend uniquely on P_1 , Q_1 and μ_1 for all $n > 1$.

Proof. Note that \tilde{H} is invariant under the transformation $(x, y) \mapsto (x, -y)$. This implies that the vertical separatrix is symmetric under reflection with respect to the x -axis. So we choose a power series with each degree having the first component even and the second odd.

To solve Hamilton's equations we use the fact that $\dot{\sigma} = \frac{1}{2}\varepsilon(1 - \sigma^2)$. Then it is a matter of substitution and gathering of terms in increasing degrees of ε .

The first term that appear in Hamilton's equations is of order 2, let $P_1(\sigma) = A_{1,0}$ and $Q_1(\sigma) = A_{1,1}\sigma$. Then we have

$$\begin{aligned}0 &= 2b_{0,0}A_{1,0}A_{1,1}\sigma\varepsilon^2 - 2a_{0,1}\mu_1A_{1,1}\sigma a_{0,1}\mu_1\varepsilon^2 \\ \frac{1}{2}A_{1,1}(1 - \sigma^2)\varepsilon^2 &= b_{0,0}(A_{1,0}^2 - A_{1,1}^2\sigma^2)\varepsilon^2 + 2a_{0,1}\mu_1A_{1,0}\varepsilon^2\end{aligned}$$

and from the possible solutions we choose $A_{1,0} = \frac{1}{2\sqrt{3}b_{0,0}}$, $A_{1,1} = \frac{1}{2b_{0,0}}$ and $\mu_1 = \frac{1}{2\sqrt{3}a_{0,1}}$.

³Here the dot denotes derivation with respect to τ .

Then we let

$$P_n(\sigma) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} A_{n,2k} \sigma^{2k},$$

$$Q_n(\sigma) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} A_{n,2k+1} \sigma^{2k+1}.$$

Thus for each n there are $n+2$ coefficients, counting μ_n as an unknown. By taking into the account that at the power ϵ^n , P_n and Q_n appear only in the second order terms of the Hamiltonian equations, we find that we need to solve a linear system. We have two cases.

- $n = 2m$

We arrange the unknowns by $(\mu_{2m}, A_{2m,1}, \dots, A_{2m,2m-1}, A_{2m,0}, \dots, A_{2m,2m})$. Then the matrix, M , of the system is of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D being $(m+1) \times (m+1)$ matrices:

$$A = \begin{pmatrix} d_0 & t_1 & & & & \\ & d_1 & t_2 & & & \\ & & d_2 & t_3 & & \\ & & & \ddots & \ddots & \\ & & & & d_{m-1} & t_m \\ & & & & & d_m \end{pmatrix},$$

$$B = -\frac{2}{\sqrt{3}} \text{Id}_{n+1},$$

$$C = \begin{pmatrix} \frac{a_{0,1}}{b_{0,0}} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} -1 & 1 & & & & \\ & -2 & 2 & & & \\ & & -3 & 3 & & \\ & & & \ddots & \ddots & \\ & & & & -m & m \\ & & & & & -m-1 \end{pmatrix},$$

with $d_0 = \frac{a_{0,1}}{\sqrt{3}b_{0,0}}$ and for $j > 0$ $d_j = \frac{1}{2} - j$, $t_j = \frac{1}{2} + j$.

- $n = 2m + 1$

We arrange the unknowns by $(\mu_{2m+1}, A_{2m+1,1}, \dots, A_{2m+1,2m+1}, A_{2m+1,0}, \dots, A_{2m+1,2m})$. Then the matrix, M , of the system has a similar form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $(m+2) \times (m+2)$, $(m+2) \times (m+1)$, $(m+1) \times (m+2)$ and $(m+1) \times (m+1)$ matrices respectively and

$$A = \begin{pmatrix} d_0 & t_1 & & & \\ & d_1 & t_2 & & \\ & & d_2 & t_3 & \\ & & & \ddots & \ddots \\ & & & & d_m & t_{m+1} \\ & & & & & d_{m+1} \end{pmatrix},$$

$$B = \begin{pmatrix} -\frac{2}{\sqrt{3}} \text{Id}_{n+1} \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{a_{0,1}}{b_{0,0}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 1 & & & & \\ & -2 & 2 & & & \\ & & -3 & 3 & & \\ & & & \ddots & \ddots & \\ & & & & -m & m \\ & & & & & -m-1 \end{pmatrix},$$

Then we have $\det(M) = \det(A - BD^{-1}C) \det(D)$. Since

$$D^{-1} = - \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m} & \frac{1}{m+1} \\ & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m} & \frac{1}{m+1} \\ & & \frac{1}{3} & \cdots & \frac{1}{m} & \frac{1}{m+1} \\ & & & \ddots & \vdots & \vdots \\ & & & & \frac{1}{m} & \frac{1}{m+1} \\ & & & & & \frac{1}{m+1} \end{pmatrix}$$

we get

$$BD^{-1}C = \begin{pmatrix} \frac{2a_{0,1}}{\sqrt{3}b_{0,0}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

so $\det(A - BD^{-1}C) \neq 0$. This means that the matrix is invertible so the system is

solvable. \square

Lemma 4.4. *The formal solution $\tilde{\mathcal{W}}(\varepsilon; \tau)$ satisfies $\tilde{\mathcal{W}}(\varepsilon; \tau + 1) = \tilde{F}_\varepsilon(\tilde{\mathcal{W}}(\varepsilon; \tau))$.*

Proof. Let \mathcal{H} be a formal series in x and y and \mathcal{X} a formal series in τ and let $T : (\mathcal{H}, \mathcal{X}) \mapsto \mathcal{X}(\tau + 1) - \phi_{\mathcal{H}}^1(\mathcal{X}(\tau))$. Then T as a map from formal series to formal series is continuous in the topology described in section 2.4. Let \tilde{H}_n be the truncation of $\tilde{H}(\mu(\varepsilon); x, y)$ to power n and $\tilde{\mathcal{W}}_{\tilde{H}_n}$ the separatrix. Since \tilde{H}_n is a polynomial, $\tilde{\mathcal{W}}_{\tilde{H}_n}$ is convergent and then $T(\tilde{H}_n, \tilde{\mathcal{W}}_{\tilde{H}_n}) = 0$. Then taking the limit $n \rightarrow \infty$ we get $T(\tilde{H}(\mu(\varepsilon); x, y), \tilde{\mathcal{W}}) = 0$ by continuity. \square

We denote $\mathcal{Z}_n(\tau) = (P_n(\sigma), Q_n(\sigma))$, so $\tilde{\mathcal{W}}(\varepsilon; \tau) = \sum_{n \geq 1} \varepsilon^n \mathcal{Z}_n(\sigma)$, and $\tilde{\mathcal{Z}}_n(\varepsilon; \tau) = \sum_{m=1}^n \varepsilon^m \mathcal{Z}_m(\sigma)$.

Corollary 4.5. *Let F_ε be a map that agrees with \tilde{F}_ε up to degree n . Then we have $\tilde{\mathcal{Z}}_n(\varepsilon; \tau + 1) - F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; \tau)) = O(\varepsilon^{n+2})$.*

Proof. This is derived directly from the continuity of the map T defined above. \square

4.4.1 Approximation of the separatrix

For the existence the two separatrices we have the following theorem.

Theorem 4.6. *Let $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$ and let $\Gamma_\varepsilon(\mathbf{x}) = \mathbf{x} + \varepsilon H_\varepsilon(\mathbf{x})$ denote a real analytic family of area preserving maps, that is also analytic in ε , defined on a bounded domain $\mathcal{D} \subset \mathbb{C}^2$ for all ε . Moreover let the origin be a hyperbolic fixed point for every map and ε be the logarithm of the largest eigenvalue. We consider the separatrix equation*

$$\mathbf{X}^-(\varepsilon; s + \varepsilon) = \Gamma_\varepsilon(\mathbf{X}^-(\varepsilon; s)).$$

Then the following are true.

- *The separatrix equation has a solution tangent to the eigenvector of $\Gamma'_\varepsilon(0)$ that corresponds to the eigenvalue that is bigger than 1.*
- *There exists a formal solution of the separatrix equation of the form*

$$\tilde{\mathbf{X}}(\varepsilon; s) = \sum_{k \geq 0} \varepsilon^k \Psi_k(e^s),$$

with Ψ_k being analytic around 0 and $\Psi_k(0) = 0$.

- Let $\tilde{\mathbf{X}}_n(\varepsilon; s) = \sum_{k=0}^{n-1} \varepsilon^k \Psi_k(e^s)$. Then we have

$$\left| \mathbf{X}^-(\varepsilon; s) - \tilde{\mathbf{X}}_n(\varepsilon; s) \right| \leq C_n \varepsilon^n$$

for all $s \in D$, where D is the domain on which all Ψ_k are bounded.

For a proof of this theorem see [BG08].

In order to apply the theorem we need to scale and translate the map. Let εw_* be one equilibrium point, namely $\varepsilon w_* = F_\varepsilon(\varepsilon w_*)$. We define the map

$$G_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon} F_\varepsilon(\varepsilon(\mathbf{x} + w_*)) - w_*.$$

We see that $G_\varepsilon(0) = 0$ and that $\mathbf{X}^-(\varepsilon; s) = \frac{1}{\varepsilon} W^-(\varepsilon; \frac{s}{\varepsilon}) - w_*$ satisfies both

$$\begin{aligned} \mathbf{X}^-(\varepsilon; s + \varepsilon) &= G_\varepsilon(\mathbf{X}^-(\varepsilon; s)) \\ \lim_{s \rightarrow -\infty} \mathbf{X}^-(\varepsilon; s) &= 0. \end{aligned}$$

Moreover by defining $\tilde{G}_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon} \tilde{F}_\varepsilon(\varepsilon(\mathbf{x} + w_*)) - w_*$, we see that we get a formal solution of the separatrix by defining

$$\begin{aligned} \tilde{\mathbf{X}}(\varepsilon; s) &= \frac{1}{\varepsilon} \tilde{W}(\varepsilon; \frac{s}{\varepsilon}) = \sum_{n \geq 1} \varepsilon^{n-1} \mathcal{Z}_n \left(\tanh \left(\frac{s}{2} \right) \right) - w_* \\ &= \sum_{n \geq 1} \varepsilon^{n-1} \mathcal{Z}_n \left(\frac{e^s - 1}{e^s + 1} \right) - w_*. \end{aligned}$$

We know that in our case the asymptotic can be written as a series of polynomials in $\tanh(s/2)$, so let D be a domain where $\tanh(s/2)$ is bounded. This means that each term of the asymptotic is bounded. From the above we see that if F_ε agrees with \tilde{F}_ε at least up to degree $n + 1$ we can apply the theorem and translate the result back to our original setting to get that there exists $C_n > 0$ such that for all $t \in D$ it holds

$$\left| W^-(\varepsilon; \tau) - \tilde{\mathcal{Z}}_n(\varepsilon; \tau) \right| \leq C_n \varepsilon^{n+1}.$$

4.4.2 Formal separatrix close to the singularity

We saw, using Theorem 4.6, that there exists a formal solution for the separatrix equation and it can be made to agree with $\tilde{\mathcal{W}}$ up to any order. Let \tilde{W} denote

	$\tilde{\mathcal{W}}_0$	$\varepsilon \tilde{\mathcal{W}}_1$	$\varepsilon^2 \tilde{\mathcal{W}}_2$	$\varepsilon^3 \tilde{\mathcal{W}}_3$	$\varepsilon^4 \tilde{\mathcal{W}}_4$	$\varepsilon^5 \tilde{\mathcal{W}}_5$	$\varepsilon^6 \tilde{\mathcal{W}}_6$	$\varepsilon^7 \tilde{\mathcal{W}}_7$	$\varepsilon^8 \tilde{\mathcal{W}}_8$	\dots
$\varepsilon \mathcal{Z}_1$	t^{-1}	ε	$\varepsilon^2 t$	$\varepsilon^3 t^2$	$\varepsilon^4 t^3$	$\varepsilon^5 t^4$	$\varepsilon^6 t^5$	$\varepsilon^7 t^6$	$\varepsilon^8 t^7$	\dots
$\varepsilon^2 \mathcal{Z}_2$	t^{-2}	εt^{-1}	ε^2	$\varepsilon^3 t$	$\varepsilon^4 t^2$	$\varepsilon^5 t^3$	$\varepsilon^6 t^4$	$\varepsilon^7 t^5$	$\varepsilon^8 t^6$	\dots
$\varepsilon^3 \mathcal{Z}_3$	t^{-3}	εt^{-2}	$\varepsilon^2 t^{-1}$	ε^3	$\varepsilon^4 t$	$\varepsilon^5 t^2$	$\varepsilon^6 t^3$	$\varepsilon^7 t^4$	$\varepsilon^8 t^5$	\dots
$\varepsilon^4 \mathcal{Z}_4$	t^{-4}	εt^{-3}	$\varepsilon^2 t^{-2}$	$\varepsilon^3 t^{-1}$	ε^4	$\varepsilon^5 t$	$\varepsilon^6 t^2$	$\varepsilon^7 t^3$	$\varepsilon^8 t^4$	\dots
$\varepsilon^5 \mathcal{Z}_5$	t^{-5}	εt^{-4}	$\varepsilon^2 t^{-3}$	$\varepsilon^3 t^{-2}$	$\varepsilon^4 t^{-1}$	ε^5	$\varepsilon^6 t$	$\varepsilon^7 t^2$	$\varepsilon^8 t^3$	\dots
$\varepsilon^6 \mathcal{Z}_6$	t^{-6}	εt^{-5}	$\varepsilon^2 t^{-4}$	$\varepsilon^3 t^{-3}$	$\varepsilon^4 t^{-2}$	$\varepsilon^5 t^{-1}$	ε^6	$\varepsilon^7 t$	$\varepsilon^8 t^2$	\dots
$\varepsilon^7 \mathcal{Z}_7$	t^{-7}	εt^{-6}	$\varepsilon^2 t^{-5}$	$\varepsilon^3 t^{-4}$	$\varepsilon^4 t^{-3}$	$\varepsilon^5 t^{-2}$	$\varepsilon^6 t^{-1}$	ε^7	$\varepsilon^8 t$	\dots
$\varepsilon^8 \mathcal{Z}_8$	t^{-8}	εt^{-7}	$\varepsilon^2 t^{-6}$	$\varepsilon^3 t^{-5}$	$\varepsilon^4 t^{-4}$	$\varepsilon^5 t^{-3}$	$\varepsilon^6 t^{-2}$	$\varepsilon^7 t^{-1}$	ε^8	\dots
$\varepsilon^9 \mathcal{Z}_9$	t^{-9}	εt^{-8}	$\varepsilon^2 t^{-7}$	$\varepsilon^3 t^{-6}$	$\varepsilon^4 t^{-5}$	$\varepsilon^5 t^{-4}$	$\varepsilon^6 t^{-3}$	$\varepsilon^7 t^{-2}$	$\varepsilon^8 t^{-1}$	\dots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	

Table 4.1: Monomials in expansion close to the singularity

this formal solution, i.e. \tilde{W} satisfies formally $\tilde{W}(\varepsilon; \tau + 1) = F_\varepsilon(\tilde{W}(\varepsilon; \tau))$. The only practical difference between \tilde{W} and $\tilde{\mathcal{W}}$ is that \tilde{W} does not have one even and one odd component. Since \tilde{W} is constructed inductively we deduce that in every order of ε is an analytic function of $\tanh(\varepsilon\tau)/2$.

Both \tilde{W} and $\tilde{\mathcal{W}}$ have a singularity at $\pi i/\varepsilon$ as the hyperbolic tangent has a simple pole there. We introduce a new parameter t by translating the origin at the singularity, so $\tau = t + \pi i/\varepsilon$. Now we can take the Laurent series around the origin. Since the power of σ in P_n and Q_n is at most n , the expansion has terms with non-negative exponents of ε . This expansion will give a formal series in t and ε with monomials summarized in table 4.1. Both \tilde{W} and $\tilde{\mathcal{W}}$ have components that close to $\pi i/\varepsilon$ expand in power series with these monomials.

We expand $\varepsilon^n \mathcal{Z}_n(\varepsilon; t + \frac{\pi i}{\varepsilon}) = \sum_{k \geq 0} W_{n,k} \varepsilon^k t^{k-n}$. This denotes both components, so $W_{n,k}$ should be thought of as a point in \mathbb{C}^2 . On Table 4.1 each row shows the monomials in the expansion of $\varepsilon^n \mathcal{Z}_n$ without the coefficients. By changing summation order we can sum by columns so we have

$$\tilde{W}(\varepsilon; t + \frac{\pi i}{\varepsilon}) = \sum_{n \geq 0} \varepsilon^n \tilde{\mathcal{W}}_n(t),$$

with each $\tilde{\mathcal{W}}_n(t)$ being a 2-vector of formal series in t . From now on $\tilde{\mathfrak{W}}(\varepsilon; t)$ will denote $\tilde{W}(\varepsilon; t + \frac{\pi i}{\varepsilon})$ summed by columns. We substitute that in the equation $\tilde{\mathfrak{W}}(\varepsilon; t +$

1) = $F_\varepsilon(\tilde{\mathfrak{W}}(\varepsilon; t))$ and we gather terms in powers of ε and we get

$$\begin{aligned} n = 0 : \quad & \tilde{\mathcal{W}}_0(t+1) = F_0(\tilde{\mathcal{W}}_0(t)) \\ n > 0 : \quad & \tilde{\mathcal{W}}_n(t+1) = F'_0(\tilde{\mathcal{W}}_0(t)) \cdot \tilde{\mathcal{W}}_n(t) + A_n(t), \end{aligned}$$

with $A_n(t)$ depending on $\tilde{\mathcal{W}}_m$ and F_m , $0 \leq m < n$.

The series $\tilde{\mathcal{W}}_0(t)$ solves equation (3.3) and it is actually the series defined in Lemma 3.2, whose Borel transform we already know.

For $n > 0$ we can find the Borel-Laplace sum by solving a non-homogeneous linear equation as in 3.3.2 by noting that $A_n(t)$ is a resurgent function as it is a substitution of resurgent functions in a convergent series. A proof can be found in [Sau15]. We saw that the solution of such equations is not defined uniquely. However by matching the solution with the asymptotic we get uniqueness. Also since there are no logarithmic terms in the expansion of $\tilde{\mathcal{W}}$ there are no logarithmic terms in the expansion of \tilde{W} . Existence of logarithmic terms in \tilde{W} in any degree would imply existence of logarithmic terms in $\tilde{\mathcal{W}}$ as the transformation from one to the other up to a given degree is analytic and it cannot create singularities. Thus the Borel transform, $\hat{\mathcal{W}}_n$, is in the space $(\widehat{\text{RES}}^{\text{sim}})^2$, which implies that similarly to $\tilde{\mathcal{W}}_0$, there are two Borel-Laplace sums for each $\tilde{\mathcal{W}}_n$, namely \mathcal{W}_n^+ and \mathcal{W}_n^- defined on the same domains as the Borel-Laplace sums of $\tilde{\mathcal{W}}_0$ and each one is the sum of a polynomial of at most degree n and a function decaying as t^{-1} as t goes to infinity.

4.4.3 Formal solution to the variational equation

Let $\tilde{\mathfrak{W}}(\varepsilon; t)$ be the formal separatrix expanded in powers of ε and t . We define a degree of each monomial by $\deg(\varepsilon^n t^m) = 2n - m$. We know that $\tilde{\mathfrak{W}}$ satisfies the equation

$$\begin{aligned} \partial_t \tilde{\mathfrak{W}}_1(\varepsilon; t) &= \tilde{H}_y(\varepsilon; \tilde{\mathfrak{W}}_1(\varepsilon; t), \tilde{\mathfrak{W}}_2(\varepsilon; t)) \\ \partial_t \tilde{\mathfrak{W}}_2(\varepsilon; t) &= -\tilde{H}_x(\varepsilon; \tilde{\mathfrak{W}}_1(\varepsilon; t), \tilde{\mathfrak{W}}_2(\varepsilon; t)), \end{aligned}$$

with $\tilde{H}_x(\varepsilon; x, y) = \partial_x \tilde{H}(\varepsilon; x, y)$ and $\tilde{H}_y(\varepsilon; x, y) = \partial_y \tilde{H}(\varepsilon; x, y)$. We also define $\tilde{H}_{xy}(\varepsilon; x, y) = \partial_x \partial_y \tilde{H}(\varepsilon; x, y)$ and similarly $\tilde{H}_{xx}(\varepsilon; x, y)$ and $\tilde{H}_{yy}(\varepsilon; x, y)$.

We know that $\tilde{H}(\varepsilon; x, -y) = \tilde{H}(\varepsilon; x, y)$. This implies that $\tilde{H}_x(\varepsilon; x, -y) = \tilde{H}_x(\varepsilon; x, y)$, $\tilde{H}_y(\varepsilon; x, -y) = -\tilde{H}_y(\varepsilon; x, y)$, $\tilde{H}_{xx}(\varepsilon; x, -y) = \tilde{H}_{xx}(\varepsilon; x, y)$, $\tilde{H}_{xy}(\varepsilon; x, -y) = -\tilde{H}_{xy}(\varepsilon; x, y)$ and $\tilde{H}_{yy}(\varepsilon; x, -y) = \tilde{H}_{yy}(\varepsilon; x, y)$.

We also define

$$\tilde{H}_\kappa(t) := \tilde{H}_\kappa(\varepsilon; \tilde{\mathbb{W}}_1(\varepsilon; t), \tilde{\mathbb{W}}_2(\varepsilon; t))$$

with $\kappa \in \{x, y, xx, xy, yy\}$. Notice that we write just $\tilde{H}_\kappa(t)$ and the dependence on ε is implied.

Since $\tilde{\mathbb{W}}_1(\varepsilon; t)$ is even in t and $\tilde{\mathbb{W}}_2(\varepsilon; t)$ is odd in t we have that

- $\tilde{H}_x(t)$, $\tilde{H}_{xx}(t)$ and $\tilde{H}_{yy}(t)$ are even functions of t ,
- $\tilde{H}_y(t)$ and $\tilde{H}_{xy}(t)$ are odd functions of t .

Let $\tilde{\mathbb{V}}(\varepsilon; t)$ be the fundamental solution of the variation of the above equation, i.e.

$$\partial_t \tilde{\mathbb{V}}(\varepsilon; t) = \tilde{J}(t) \tilde{\mathbb{V}}(\varepsilon; t)$$

with

$$\tilde{J}(t) = \begin{pmatrix} \tilde{H}_{xy}(t) & \tilde{H}_{yy}(t) \\ -\tilde{H}_{xx}(t) & -\tilde{H}_{xy}(t) \end{pmatrix}.$$

Let $\tilde{\mathbb{V}}(\varepsilon; t) = (\tilde{\Xi}(\varepsilon; t), \tilde{Z}(\varepsilon; t))$. We ask that $\det \tilde{\mathbb{V}}(\varepsilon; t) = 1$ and that $\tilde{Z}(\varepsilon; t) = \partial_t \tilde{\mathbb{W}}(\varepsilon; t)$. We write

$$\tilde{\Xi}_1(\varepsilon; t) = \frac{1 + \tilde{\Xi}_2(\varepsilon; t) \tilde{Z}_1(\varepsilon; t)}{\tilde{Z}_2(\varepsilon; t)}$$

and we substitute this in the variational equation to get

$$\partial_t \tilde{\Xi}_2(\varepsilon; t) = - \left(\tilde{H}_{xx}(t) \frac{\tilde{Z}_1(\varepsilon; t)}{\tilde{Z}_2(\varepsilon; t)} + \tilde{H}_{xy}(t) \right) \tilde{\Xi}_2(\varepsilon; t) - \frac{\tilde{H}_{xx}(t)}{\tilde{Z}_2(\varepsilon; t)}.$$

Since $\tilde{Z}_2(\varepsilon; t)$ is a solution for the homogeneous equation we write $\tilde{\Xi}_2(\varepsilon; t) = C(\varepsilon; t) \tilde{Z}_2(\varepsilon; t)$ and substitute in the previous equation to finally get

$$\partial_t C(\varepsilon; t) = - \frac{\tilde{H}_{xx}(t)}{\tilde{Z}_2(\varepsilon; t)^2}.$$

Since both $\tilde{H}_{xx}(t)$ and $\tilde{Z}_2(\varepsilon; t)$ are even series the above equation can be solved in the space of power series and $C(\varepsilon; t)$ is an odd series without logarithmic terms. This implies that $\tilde{\Xi}_2(\varepsilon; t)$ is odd and $\tilde{\Xi}_1(\varepsilon; t)$ is even.

This formal solution also satisfies the variational equation of the map, i.e.

$$\tilde{\mathbb{V}}(\varepsilon; t+1) = F'_\varepsilon(\tilde{\mathbb{W}}(\varepsilon; t))\tilde{\mathbb{V}}(\varepsilon; t).$$

We write

$$\tilde{\mathbb{V}}(\varepsilon; t) = \sum_{n \geq 0} \varepsilon^n \tilde{\mathbb{V}}_n(t)$$

and we see that $\tilde{\mathbb{V}}_0(t)$ satisfies the equation

$$\tilde{\mathbb{V}}_0(t+1) = F'_0(\tilde{\mathbb{W}}_0(t))\tilde{\mathbb{V}}_0(t).$$

By taking into account the normalization we get that

$$\tilde{\mathbb{V}}_0(t) = \tilde{\mathcal{V}}(t),$$

where $\tilde{\mathcal{V}}$ is the formal fundamental solution of the variational equation at the resonance, see Section 3.3.1.

Then for $n > 0$ we get

$$\tilde{\mathbb{V}}_n(t+1) = F'_0(\tilde{\mathbb{W}}_0(t))\tilde{\mathbb{V}}_n(t) + \tilde{B}_n(t)$$

with \tilde{B}_n depending on $\tilde{\mathbb{V}}_m$'s with $m < n$. As we saw in the previous chapter these equations define resurgent series and they can be solved in the Borel plane. Since we know that the series contain only integer powers of t , then $\hat{\mathbb{V}}_n$, the Borel transform of $\tilde{\mathbb{V}}_n$, is a simple resurgent function. By looking at the equation above we get that for $\tilde{\mathbb{V}}_n$ the biggest power of t is $n+2$.

We already know that $\tilde{\mathcal{V}}_n$ is resurgent so we can define the action of the alien derivative Δ_{ω_m} on $\tilde{\mathbb{W}}$ by

$$\Delta_{\omega_m}[\tilde{\mathbb{W}}](\varepsilon; t) = \sum_{n \geq 0} \varepsilon^n \Delta_{\omega_m}[\tilde{\mathcal{W}}_n](t).$$

Since Δ_{ω_m} satisfies the Leibniz rule,⁴ then all $\Delta_{\omega_m}[\tilde{\mathbb{W}}](\varepsilon; t)$ satisfy the variational equation, which means that for all $m \geq 1$ there are constants Θ_{ω_m} and q_{ω_m} such that

$$\Delta_{\omega_m}[\tilde{\mathbb{W}}](\varepsilon; t) = \Theta_{\omega_m} \tilde{\Xi}(\varepsilon; t) + q_{\omega_m} \tilde{Z}(\varepsilon; t).$$

⁴ This is because $\Delta_{\omega_m}[X_k Y_{n-k}](t) = \Delta_{\omega_m}[X_k](t)Y_{n-k}(t) + X_k(t)\Delta_{\omega_m}[Y_{n-k}](t)$ implies that $\Delta_{\omega_m}[XY](\varepsilon; t) = \Delta_{\omega_m}[X](\varepsilon; t)Y(\varepsilon; t) + X(\varepsilon; t)\Delta_{\omega_m}[Y](\varepsilon; t)$.

4.5 Complex matching

In this section we will show that the formal solution $\tilde{\mathfrak{W}}^\pm$ describes the asymptotic behaviour of W^\pm close to the singularity.

Let $\text{SQ}(r), \text{SQ}_{+1}(r), \text{HP}(r) \in \mathbb{C}$ be defined as follows

$$\begin{aligned}\text{SQ}(r) &:= \{z \in \mathbb{C} : |\text{Im}(z)| < r, \text{Re}(z) > -r\}, \\ \text{SQ}_{+1}(r) &:= \{z \in \mathbb{C} : |\text{Im}(z)| < r, \text{Re}(z) > -(r+1)\}, \\ \text{HP}(r) &:= \{z \in \mathbb{C} : \text{Re}(z) > r\}.\end{aligned}$$

Recall that we assume that there exists $\varepsilon_0 > 0$ such that $\varepsilon \in (0, \varepsilon_0)$. Since we are interested in the asymptotic behaviour of the separatrices, we can choose ε_0 to be as small as it is convenient. We choose $\Lambda > 1$ such that $\Lambda^2 \varepsilon_0 < 1$. During the course of this proof we will see that it may be important to increase the value of Λ . This is not a problem since we can simultaneously decrease ε_0 such that the relation $\Lambda^2 \varepsilon_0 < 1$ still holds. So Λ should be thought of as a constant but one that can be increased later if needed.

We choose Λ and fix $R > 1$ and $\Lambda > 1$ and we define the following domains:

$$\begin{aligned}D_0 &:= \mathbb{C} \setminus (\text{SQ}((\Lambda\varepsilon)^{-1}) \cup \text{HP}(R)), \\ D_1 &:= \text{SQ}_{+1}((\Lambda\varepsilon)^{-1}) \setminus (\text{SQ}(\varepsilon^{-\frac{1}{2}}) \cup \text{HP}(\Lambda)), \\ D_2 &:= \text{SQ}_{+1}(\varepsilon^{-\frac{1}{2}}) \setminus (\text{SQ}(\Lambda) \cup \text{HP}(\Lambda)).\end{aligned}$$

These can be seen in Figures 4.1. Note that D_1 intersects D_0 on a narrow strip of width 1 on the left of Figure 4.1b and that D_2 intersects D_1 on an other narrow strip of width 1.

Definition 4.7. Let $n \in \mathbb{N}$, $n \leq N$. We define

$$\begin{aligned}\tilde{\mathfrak{W}}_n^\pm(\varepsilon; t) &:= \sum_{k=0}^{n-1} \varepsilon^k \mathscr{W}_k^\pm(t), \\ \tilde{\mathcal{Z}}_n(\varepsilon; t) &:= \sum_{k=1}^n \varepsilon^k \mathcal{Z}_k(\sigma).\end{aligned}$$

The main result of this section is the following lemma.

Lemma 4.8. *There exists $\Lambda > 1$ and $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$, $5 \leq n \leq N$*

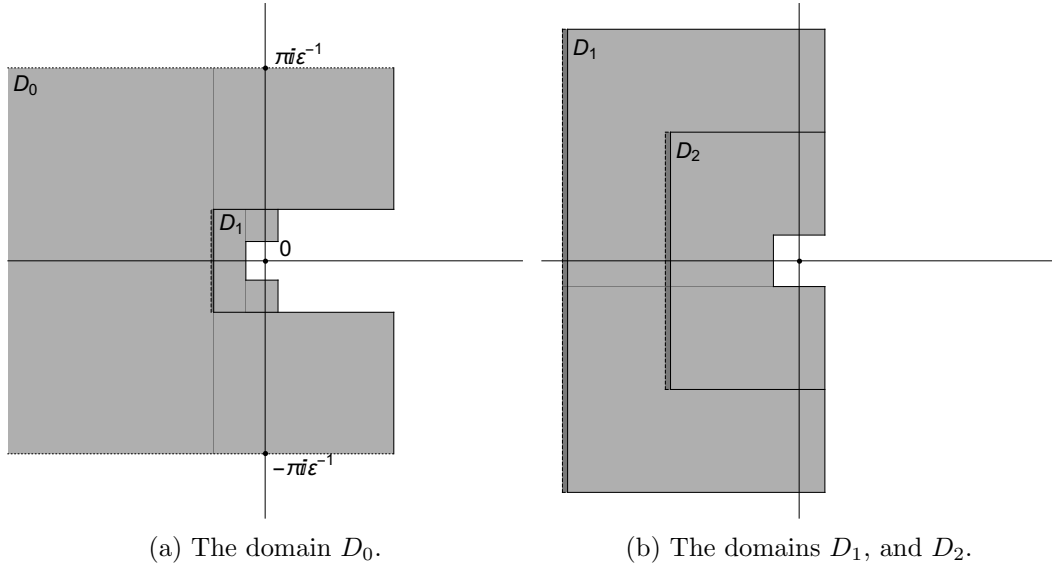


Figure 4.1: The domains considered.

and every $\varepsilon < \varepsilon_0$ there exists $C_2 > 0$ such that for all $t \in D_2$ it holds

$$\left\| W^-(\varepsilon; t + \frac{\pi}{\varepsilon} \mathbf{i}) - \tilde{\mathfrak{W}}_n^-(\varepsilon; t) \right\|_{\infty} \leq C_2 \varepsilon^{\frac{n-1}{2}}.$$

The rest of this section is devoted to the proof of this lemma.

Lemma 4.9. *Let A be a 2×2 matrix. We view A as a linear map from \mathbb{C}^2 to \mathbb{C}^2 , both equipped with the supremum norm. Then*

$$\|A\|_{\infty} = \max\{|A_{1,1}| + |A_{1,2}|, |A_{2,1}| + |A_{2,2}|\}.$$

Proof. Using the definition we get

$$\|A\|_{\infty} = \sup_{\|z\|_{\infty}=1} \|Az\|_{\infty} = \sup_{|z_1|, |z_2| \leq 1} \max\{|A_{1,1}z_1 + A_{1,2}z_2|, |A_{2,1}z_1 + A_{2,2}z_2|\}$$

and from this the result follows. \square

Lemma 4.10. *Let $Q(t) : D_1 \rightarrow \mathbb{C}^2$ and $c > 0$ such that $\|Q(t)\|_{\infty} \leq c|t|^{-2}$. Then there exist $C_{1,1}, C_{1,2} > 0$ such that*

$$\left\| F'_{\varepsilon}(\varepsilon \mathcal{Z}_1(\sigma) + Q(t)) \right\|_{\infty} \leq 1 + \frac{2}{|t|} + C_{1,1}\varepsilon + \frac{C_{1,2}}{|t|^2}.$$

Proof. Let $s \in \mathbb{C}$, if $|s| < 1/2$ it holds

$$\tanh\left(\frac{\pi i}{2} + s\right) = \frac{1}{s} + s\phi(s)$$

with

$$|\phi(s)| \leq 1$$

and

$$\left| \tanh\left(\frac{\pi i}{2} + s\right) \right| \leq \frac{2}{|s|}.$$

So

$$\varepsilon\sigma = \frac{2}{t} + \frac{\varepsilon^2 t}{2} \phi\left(\frac{\varepsilon t}{2}\right),$$

Note that \mathcal{F}'_1 is the identity and

$$\mathcal{F}'_2(x, y) = \begin{pmatrix} -2b_{0,0}y & -2b_{0,0}x + \frac{\varepsilon}{\sqrt{3}} \\ -2b_{0,0}x - \frac{\varepsilon}{\sqrt{3}} & 2b_{0,0}y \end{pmatrix}.$$

This gives

$$\mathcal{F}'_2(\varepsilon\mathcal{Z}_1(\sigma)) = \begin{pmatrix} \varepsilon\sigma & 0 \\ \frac{\sqrt{3}}{2}\varepsilon & -\varepsilon\sigma \end{pmatrix}$$

and $\|\mathcal{F}'_2(Q(t))\|_\infty \leq \frac{C_1}{|t|^2} + C_2\varepsilon$.

We have $F'_\varepsilon(x, y) = \sum_{n \geq 1} \mathcal{F}'_n(\varepsilon; x, y)$. If $t \in D_1$ we have that the first component of $\varepsilon\mathcal{Z}_1(\sigma)$ is a constant times ε and the second component is bounded by a constant over $|t|$. From this we get

$$\begin{aligned} |\varepsilon\mathcal{Z}_1(\sigma) + Q(t)| &= \frac{1}{|t|} |t\varepsilon\mathcal{Z}_1(\sigma) + tQ(t)| \\ &\leq \frac{1}{2b_{0,0}|t|} \left(\left| \begin{pmatrix} \frac{\varepsilon t}{\sqrt{3}} \\ 2 + \frac{\varepsilon^2 t^2}{2} \phi\left(\frac{\varepsilon t}{2}\right) \end{pmatrix} \right| + \left| \begin{pmatrix} \frac{2b_{0,0}c}{|t|} \\ \frac{2b_{0,0}c}{|t|} \end{pmatrix} \right| \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2b_{0,0}|t|} \left(\frac{\frac{\varepsilon|t|}{\sqrt{3}} + \frac{2b_{0,0}c}{|t|}}{\left| 2 + \frac{\varepsilon^2 t^2}{2} \phi\left(\frac{\varepsilon t}{2}\right) \right| + \frac{2b_{0,0}c}{|t|}} \right) \\
&\leq \frac{1}{2b_{0,0}|t|} \left(\frac{\frac{1}{\sqrt{3}\Lambda} + \frac{2b_{0,0}c}{\Lambda}}{2 + \frac{1}{2\Lambda^2} \left| \phi\left(\frac{\varepsilon t}{2}\right) \right| + \frac{2b_{0,0}c}{\Lambda}} \right) \\
&\leq \frac{1}{2b_{0,0}|t|} \left(\frac{\frac{1}{\sqrt{3}\Lambda} + \frac{2b_{0,0}c}{\Lambda}}{2 + \frac{1}{2\Lambda^2} + \frac{2b_{0,0}c}{\Lambda}} \right) \\
&\leq \frac{1}{2b_{0,0}|t|} \left(2 + \frac{1}{\Lambda} + \frac{2b_{0,0}c}{\Lambda} \right) = \frac{C_3}{|t|},
\end{aligned}$$

where inequality and absolute value are to be interpreted componentwise. Notice that the constant C_3 is a decreasing function of Λ .

Then we look at $\mathcal{F}'_n(\varepsilon; x, y)$ for $n \geq 3$. Each monomial of \mathcal{F}'_n is of degree $n - 1$. We substitute $\varepsilon \mathcal{Z}_1(\sigma) + Q(t)$ in $\sum_{n \geq 3} \mathcal{F}'_n$, then all the monomials that have a non-zero power of ε are in $O(\varepsilon)$ and all other are in $O(|t|^{-2})$.

Collecting everything together we get the result. Notice that the constant $C_{1,2}$ is a decreasing function of Λ . \square

Lemma 4.11. *Let $Q(t) : D_2 \rightarrow \mathbb{C}^2$ and assume that there exists $c > 0$ such that $|Q(t)| \leq c\varepsilon$. Then there exist $C_{2,1}, C_{2,2} > 0$ such that*

$$\|F'_\varepsilon(\mathcal{W}_0^-(t) + Q(t))\|_\infty \leq 1 + \frac{2}{|t|} + C_{2,1}\varepsilon + \frac{C_{2,2}}{|t|^2}.$$

Proof. We take into account that $|t| > \Lambda$. Then recall that

$$\mathcal{W}_0^-(t) = \begin{pmatrix} 0 \\ -\frac{1}{b_{0,0}t} \end{pmatrix} + r(t)$$

with $\|r(t)\|_\infty \leq C_r|t|^{-2}$, which also it implies trivially that $\|\mathcal{W}_0^-(t)\|_\infty \leq C_0|t|^{-1}$, and that

$$F_0(x, y) = \begin{pmatrix} x - 2b_{0,0}xy + b_{0,0}^2x^3 + b_{0,0}^2xy^2 \\ y - b_{0,0}x^2 + b_{0,0}y^2 + b_{0,0}^2x^2y + b_{0,0}^2y^3 \end{pmatrix} + O_4(x, y).$$

From these we get that

$$F'_0(\mathcal{W}_0^-(t) + Q(t)) = \begin{pmatrix} 1 + \frac{2}{t} & 0 \\ 0 & 1 - \frac{2}{t} \end{pmatrix} + R(t)$$

with $\|R(t)\|_\infty \leq C_R(\varepsilon + |t|^{-2})$, by Lemma 4.9. Moreover $\forall k \in \mathbb{N}$, $k \geq 1$ there exists C_k such that $\|\varepsilon^k F'_k(\mathcal{W}_0^-(t) + Q(t))\|_\infty \leq C_k \varepsilon^k \leq C_k \varepsilon \Lambda^{2-2k}$ and since F_ε is analytic around the origin we can sum and get the result. As before the constant $C_{2,2}$ is a decreasing function of Λ . \square

Lemma 4.12. *Let $\mu : \mathbb{C} \rightarrow \mathbb{R}_+$ with*

$$\mu(t) \leq 1 + \frac{2}{|t|} + c_1 \varepsilon + \frac{c_2}{|t|^2}$$

for some $c_1, c_2 > 0$. Then for all $m \in \mathbb{N}$ with $m \leq (\Lambda \varepsilon)^{-1} + 2$ it holds

$$\prod_{k=0}^m \mu(t+k) \leq C \frac{|t|^2}{|t+m|^2}$$

with

$$C = \left(1 + \frac{2}{\Lambda} + \frac{c_1 + c_2}{\Lambda^2}\right) \cdot \exp \left(2\pi + \frac{\pi}{\Lambda} \left(c_2 + \left(4 + \frac{c_1}{\Lambda} + \frac{c_2}{\Lambda}\right)^2\right) + c_1 \left(\frac{1}{\Lambda} + \frac{2}{\Lambda^2}\right)\right).$$

Proof. For all $x \in \mathbb{R}$ with $x \geq 0$, it holds $\log(1+x) = x + r(x)$ with $|r(x)| \leq x^2$. So we have

$$\begin{aligned} & \log \left(1 + 2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1 \varepsilon + \frac{c_2}{|t|^2}\right) = \\ &= 2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1 \varepsilon + \frac{c_2}{|t|^2} + r \left(\frac{1}{|t|} \left(2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|} + c_1 \varepsilon |t| + \frac{c_2}{|t|}\right)\right) \\ &\leq 2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1 \varepsilon + \frac{c_2}{|t|^2} + \frac{1}{|t|^2} \left(2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|} + c_1 \varepsilon |t| + \frac{c_2}{|t|}\right)^2 \\ &\leq 2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1 \varepsilon + \frac{1}{|t|^2} \left(c_2 + \left(4 + \frac{c_1}{\Lambda} + \frac{c_2}{\Lambda}\right)^2\right) \\ &\leq 2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1 \varepsilon + \frac{C_2}{|t|^2}. \end{aligned}$$

Then by standard integration we get

$$\begin{aligned} \int_t^{t+m} 2 \frac{|\operatorname{Re}(\mathfrak{t})| + |\operatorname{Im}(\mathfrak{t})|}{|\mathfrak{t}|^2} d\mathfrak{t} &= \log \left(\frac{|t|^2}{|t+m|^2} \right) + 2 \arctan \left(\frac{|\operatorname{Re}(t)| - m}{|\operatorname{Im}(t)|} \right) \\ &\quad - 2 \arctan \left(\frac{|\operatorname{Re}(t)|}{|\operatorname{Im}(t)|} \right) \\ &\leq \log \left(\frac{|t|^2}{|t+m|^2} \right) + 2\pi \end{aligned}$$

and

$$\int_t^{t+m} \frac{C_2}{|\mathfrak{t}|^2} d\mathfrak{t} = \frac{C_2}{|\operatorname{Im}(t)|} \arctan \left(\frac{|\operatorname{Re}(t)| - m}{|\operatorname{Im}(t)|} \right) - \frac{C_2}{|\operatorname{Im}(t)|} \arctan \left(\frac{|\operatorname{Re}(t)|}{|\operatorname{Im}(t)|} \right) \leq \frac{C_2}{\Lambda} \pi.$$

Also note that

$$c_1 m \varepsilon \leq c_1 \left(\frac{1}{\Lambda} + 2\varepsilon \right).$$

So collecting everything together we get

$$\int_t^{t+m} 2 \frac{|\operatorname{Re}(\mathfrak{t})| + |\operatorname{Im}(\mathfrak{t})|}{|\mathfrak{t}|^2} + \frac{C_2}{|\mathfrak{t}|^2} d\mathfrak{t} + c_1 \varepsilon m \leq \log \left(\frac{|t|^2}{|t+m|^2} \right) + 2\pi + \frac{C_2}{\Lambda} \pi + c_1 \left(\frac{1}{\Lambda} + 2\varepsilon \right),$$

By the above we get

$$\begin{aligned} \log \left(\prod_{k=1}^m \mu(t+k) \right) &= \sum_{k=0}^{m-1} \log(\mu(t+k)) \\ &\leq \sum_{k=1}^m 2 \frac{|\operatorname{Re}(t+k)| + |\operatorname{Im}(t+k)|}{|t+k|^2} + \frac{C_2}{|t+k|^2} + c_1 \varepsilon \\ &\leq \int_t^{t+m} 2 \frac{|\operatorname{Re}(\mathfrak{t})| + |\operatorname{Im}(\mathfrak{t})|}{|\mathfrak{t}|^2} + \frac{C_2}{|\mathfrak{t}|^2} d\mathfrak{t} + c_1 \varepsilon m \\ &\leq \log \left(\frac{|t|^2}{|t+m|^2} \right) + 2\pi + \frac{C_2}{\Lambda} \pi + c_1 \left(\frac{1}{\Lambda} + 2\varepsilon \right). \end{aligned}$$

Note that trivially $\mu(t) \leq 1 + \frac{2}{\Lambda} + \frac{c_1+c_2}{\Lambda^2}$. Then exponentiation of the last relation and multiplication by the bound of $\mu(t)$ gives the result. \square

Lemma 4.13. *There exists $\Lambda > 1$ and $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$, $5 \leq n \leq N$ and every $\varepsilon < \varepsilon_0$ there exists $C_1 > 0$ such that for all $t \in D_1$ it holds*

$$\left\| W^-(\varepsilon; t + \frac{\pi}{\varepsilon} \mathfrak{i}) - \tilde{\mathcal{Z}}_n(\varepsilon; t) \right\|_{\infty} \leq \frac{C_1}{|t|^{n+1}}.$$

Proof. Let

$$\begin{aligned}\xi_n(\varepsilon; t) &:= W^-(\varepsilon; t) - \tilde{\mathcal{Z}}_n(\varepsilon; t), \\ R_n(\varepsilon; t) &:= \tilde{\mathcal{Z}}_n(\varepsilon; t) - F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t - 1)).\end{aligned}$$

It holds $\tilde{\mathcal{Z}}_n(\varepsilon; t + 1) - F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t)) = O(\varepsilon^{n+2}\sigma^{n+2})$. It can be easily checked that $\tilde{\mathcal{Z}}_1(\varepsilon; t + 1) - F_\varepsilon(\tilde{\mathcal{Z}}_1(\varepsilon; t)) = O(\varepsilon^3\sigma^3)$ and then each order in $\tilde{\mathcal{Z}}_n$ cancels an order of the difference. So for all $t \in D_1$ it holds $R_n(\varepsilon; t) = O(|t|^{-n-2})$.

Substituting in $W^-(\varepsilon; t + 1) = F_\varepsilon(W^-(\varepsilon; t))$ we get

$$\xi_n(\varepsilon; t + 1) = \left(\int_0^1 F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t) + \mathfrak{t}\xi_n(\varepsilon; t)) d\mathfrak{t} \right) \xi_n(\varepsilon; t) + R_n(\varepsilon; t + 1),$$

from which we get

$$\xi_n(\varepsilon; t + k + 1) = \left(\int_0^1 F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t + k) + \mathfrak{t}\xi_n(\varepsilon; t + k)) d\mathfrak{t} \right) \xi_n(\varepsilon; t + k) + R_n(\varepsilon; t + k + 1).$$

Let

$$\begin{aligned}\delta_k &:= \|\xi_n(\varepsilon; t + k)\|_\infty, \\ \alpha_k &:= \left\| \int_0^1 F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t + k) + \mathfrak{t}\xi_n(\varepsilon; t + k)) d\mathfrak{t} \right\|_\infty, \\ \beta_k &:= \|R_n(\varepsilon; t + k + 1)\|_\infty.\end{aligned}$$

Then we have

$$\delta_{k+1} \leq \alpha_k \delta_k + \beta_k$$

and from this we get that

$$\delta_k \leq \left(\prod_{i=1}^{n-1} \alpha_i \right) \delta_0 + \sum_{i=0}^{k-1} \left(\prod_{j=i+1}^{k-1} \alpha_j \right) \beta_i$$

For all $t \in D_0 \cap D_1$ it holds

$$\left| W^-(\varepsilon; t) - \tilde{\mathcal{Z}}_n(\varepsilon; t) \right| \leq \frac{C_0}{|t|^{n+1}},$$

so we get $\delta_0 \leq C_0|t|^{-n-1}$ and $\beta_k \leq C_\beta|t + k + 1|^{-n-2}$ from Taylor's theorem.

Assume that there exists $C_1 > \exp(2\pi + 1)(C_0 + C_\beta)$ such that $\forall j < k$ it holds $\delta_j \leq C_1 |t + j|^{-n-1}$. Then using Lemma 4.10 we get that

$$\alpha_j \leq 1 + \frac{2}{|t + j|} + C_{1,1}\varepsilon + \frac{C_{1,2}}{|t + j|^2}$$

and

$$\begin{aligned} \delta_k &\leq \left(\prod_{i=1}^{k-1} \alpha_i \right) \delta_0 + \sum_{i=0}^{k-1} \left(\prod_{j=i+1}^{k-1} \alpha_j \right) \beta_i \\ &\leq C \frac{|t|^2}{|t + k|^2} \frac{C_0}{|t|^{n+1}} + \sum_{i=0}^{k-1} C \frac{|t + i + 1|^2}{|t + k|^2} \frac{C_\beta}{|t + i + 1|^{n+2}} \\ &\leq \frac{C_0 C}{|t + k|^2 |t|^{n-1}} + \frac{C_\beta C}{|t + k|^2} \sum_{i=0}^{k-1} \frac{1}{|t + j + 1|^n} \\ &\leq \frac{C_0 C}{|t + k|^{n+1}} + \frac{C_\beta C}{|t + k|^2} \frac{1}{|t + k|^{n-1}} \\ &\leq \frac{C(C_0 + C_\beta)}{|t + k|^{n+1}} \end{aligned}$$

We choose Λ big enough to have⁵ $C(C_0 + C_\beta) < C_1$. Then we get that the inductive hypothesis holds also for $m + 1$. This actually proves that the bound is true in $\text{SQ}_{+1}(\Lambda) \setminus \text{HP}(0)$. Of course the bound becomes arbitrarily big close to the origin so it will be used only in D_1 . To extend the bound to the whole D_1 we need to apply the same technique for Λ more steps which changes only the constants. \square

Proof of Lemma 4.8. Let

$$\begin{aligned} \xi_n(\varepsilon; t) &:= W^-(\varepsilon; t) - \tilde{\mathfrak{W}}_n^-(\varepsilon; t), \\ R_n(\varepsilon; t) &:= \tilde{\mathfrak{W}}_n^-(\varepsilon; t) - F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t - 1)). \end{aligned}$$

It holds $\tilde{\mathfrak{W}}_n^-(\varepsilon; t + 1) - F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t)) = O(\varepsilon^{n+1} t^{n-1})$. It can be easily checked that $\tilde{\mathfrak{W}}_0^-(\varepsilon; t + 1) - F_\varepsilon(\tilde{\mathfrak{W}}_0^-(\varepsilon; t)) = O(\varepsilon t^{-1})$ and then each order in $\tilde{\mathfrak{W}}_n^-$ cancels an order of the difference. So for all $t \in D_2$ it holds $R_n(\varepsilon; t) = O(\varepsilon^{n+1} t^{n-1})$.

Substituting in $W^-(\varepsilon; t + 1) = F_\varepsilon(W^-(\varepsilon; t))$ we get

$$\xi_n(\varepsilon; t + 1) = \left(\int_0^1 F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t) + \mathfrak{t} \xi_n(\varepsilon; t)) d\mathfrak{t} \right) \xi_n(\varepsilon; t) + R_n(\varepsilon; t + 1),$$

⁵ This can always be done since it is equivalent to $C \leq \exp(2\pi + 1)$.

from which we get

$$\xi_n(\varepsilon; t + k + 1) = \left(\int_0^1 F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t + k) + \mathfrak{k} \xi_n(\varepsilon; t + k)) d\mathfrak{k} \right) \xi_n(\varepsilon; t + k) + R_n(\varepsilon; t + k + 1).$$

Similarly to the above proof we define

$$\begin{aligned} \delta_k &:= \|\xi_n(\varepsilon; t + k)\|_\infty, \\ \alpha_k &:= \left\| \int_0^1 F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t + k) + \mathfrak{k} \xi_n(\varepsilon; t + k)) d\mathfrak{k} \right\|_\infty, \\ \beta_k &:= \|R_n(\varepsilon; t + k + 1)\|_\infty, \end{aligned}$$

Then again we have

$$\delta_{k+1} \leq \alpha_k \delta_k + \beta_k$$

and

$$\delta_k \leq \left(\prod_{i=1}^{n-1} \alpha_i \right) \delta_0 + \sum_{i=0}^{k-1} \left(\prod_{j=i+1}^{k-1} \alpha_j \right) \beta_i.$$

From now on we assume that $t \in D_1 \cap D_2$ and for such t it holds

$$\left\| W^-(\varepsilon; t) - \tilde{\mathfrak{W}}_n^-(\varepsilon; t) \right\|_\infty \leq \frac{C_1}{|t|^{n+1}} \leq C_1 \varepsilon^{\frac{n+1}{2}},$$

Assume that there exists $C_2 > \exp(2\pi + 1)$ such that $\forall j < k$ it holds $\delta_j \leq C_2 \varepsilon^{\frac{n-1}{2}}$.

Then using Lemma 4.11 we get that

$$\alpha_j \leq 1 + \frac{2}{|t + j|} + C_{1,1} \varepsilon + \frac{C_{1,2}}{|t + j|^2}.$$

and

$$\begin{aligned} \delta_k &\leq \left(\prod_{i=1}^{k-1} \alpha_i \right) \delta_0 + \sum_{i=0}^{k-1} \left(\prod_{j=i+1}^{k-1} \alpha_j \right) \beta_i \\ &\leq C \frac{|t|^2}{|t + k|^2} \frac{C_1}{|t|^{n+1}} + \sum_{i=0}^{k-1} C \frac{|t + j + 1|^2}{|t + k|^2} C_\beta \varepsilon^{n+1} |t + j + 1|^{n-1} \\ &\leq \frac{C_1 C}{|t + k|^2 |t|^{n-1}} + \frac{C_\beta C \varepsilon^{n+1}}{|t + k|^2} \sum_{i=0}^{k-1} |t + j + 1|^{n+1} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C_0 C}{|t+k|^2} \varepsilon^{\frac{n-1}{2}} + \frac{C_\beta C \varepsilon^{n+1}}{|t+k|^2} |t+k|^{n+2} \\
&\leq \frac{C_0 C}{|t+k|^2} \varepsilon^{\frac{n-1}{2}} + C_\beta C \varepsilon^{n+1} |t+k|^{n+1} \\
&\leq \frac{C_0 C}{|t+k|^2} \varepsilon^{\frac{n-1}{2}} + C_\beta C \varepsilon^{n+1} \left(\frac{\sqrt{2}}{\sqrt{\varepsilon}} \right)^{n+1} \\
&\leq \frac{C_0 C}{|t+k|^2} \varepsilon^{\frac{n-1}{2}} + 2^{\frac{n+1}{2}} C_\beta C \varepsilon^{\frac{n+1}{2}} \\
&\leq C \left(\frac{C_0}{|t+k|^2} + 2^{\frac{n+1}{2}} C_\beta \varepsilon \right) \varepsilon^{\frac{n-1}{2}} \\
&\leq C \left(\frac{C_0}{\Lambda^2} + 2^{\frac{n+1}{2}} \frac{C_\beta}{\Lambda^2} \right) \varepsilon^{\frac{n-1}{2}}
\end{aligned}$$

Similarly to the previous proof we can choose Λ big enough to get $\delta_k \leq C_2 \varepsilon^{\frac{n-1}{2}}$. Then by induction we get the result. \square

Using the inverse map we arrive to a similar result for the stable separatrix.

4.6 Variational equations

There are two variational equations that are very important in this analysis. In this section we will show that the solutions of both can be approximated by the same formal series.

4.6.1 Linear difference equations in a rectangular domain

We consider rectangular symmetric domains around the origin, i.e. there exist $\alpha, \beta > 1$ such that $D = \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq \alpha, |\operatorname{Im}(z)| \leq \beta\}$. Let $\mathcal{O}(D)$ be the space of functions analytic in the interior of D and continuous at its boundary with the supremum norm over D .

Let $g \in \mathcal{O}(D)$. We will examine the equation

$$X(z+1) - X(z) = g(z). \quad (4.3)$$

We define the operator

$$\mathcal{S} : X(z) \mapsto X(z+1) - X(z).$$

To solve the equation (4.3) we need to inverse the operator \mathcal{S} . We can construct the following two formal solutions

$$S^+[g](z) := - \sum_{n \geq 0} g(z+n)$$

and

$$S^-[g](z) := \sum_{n \geq 1} g(z-n).$$

Since g is defined in a compact set around the origin, the above solutions have no analytic meaning unless g can be extended beyond its initial domain of definition. Towards this end we have the following lemma.

Lemma 4.14. *Let $h \in \mathcal{O}(D)$, χ be a Lipschitz continuous function of $\partial\mathcal{D}$ and*

$$J_h = \frac{1}{2\pi} \int_{\partial\mathcal{D}} |h(\zeta)| |d\zeta| < \infty.$$

Then the integral

$$H(z) = \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{h(\zeta)\chi(\zeta)}{\zeta - z} d\zeta$$

defines two functions H_{int} and H_{ext} in the interior and the exterior of \mathcal{D} respectively. Both functions admit continuous extensions onto the closure of their respective domains and

$$|H_{int,ext}| \leq (J_h + \|h\|_\infty) \|\chi\|_{Lip}.$$

If $\text{supp}(\chi) \neq \partial\mathcal{D}$ then H_{int} and H_{ext} define a single analytic function on $\mathbb{C} \setminus \text{supp}(\chi)$.

Moreover let \mathcal{D} be contained in a square of side R . Then

$$|H_{int,ext}| \leq C \log(R) \|h\|_\infty \|\chi\|_{Lip}$$

for some $C > 0$.

For a proof see §9 in [Gel99].

We define the function $\chi^+ : \partial D \rightarrow [0, 1]$ to be Lipschitz continuous. We also ask that χ^+ has the value 1 on $\partial D \cap \{z \in \mathbb{C} : \text{Re}(z) < -\alpha/2\}$ and χ^+ has the value 0 on $\partial D \cap \{z \in \mathbb{C} : \text{Re}(z) > \alpha/2\}$. We also define $\chi^-(z) = 1 - \chi^+(z)$, which implies

that $\|\chi^+\|_{\text{Lip}} = \|\chi^-\|_{\text{Lip}} = L$. We define

$$h^\pm(z) = \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{h(\zeta)\chi^\pm(\zeta)}{\zeta - z} d\zeta.$$

The functions h^+ and h^- are analytic on $\mathbb{C} \setminus \text{supp}(\chi^+)$ and $\mathbb{C} \setminus \text{supp}(\chi^-)$ respectively and $h^+(z) + h^-(z) = h(z)$ when $z \in \mathring{D}$ because of the Cauchy integral.

With these we define

$$\mathcal{S}^{-1}[h](z) = S[h](z) := \sum_{n \geq 1} h^-(z - n) - \sum_{n \geq 0} h^+(z + n).$$

This solves equation $X(z + 1) - X(z) = h(z)$ if both sums are convergent.

In order to generalize this method we need to introduce a weigh function. Let $\phi_a(z) = e^{az} + e^{-az}$ for some $a > 0$ and we denote $\|\phi_a\|_D = \sup_{z \in D} |\phi_a(z)|$. Then we repeat the above construction with $h(z) = \phi_a(z)g(z)$. We define

$$g_a^\pm(z) = \frac{1}{2\pi i \phi_a(z)} \int_{\partial\mathcal{D}} \frac{\phi_a(\zeta)h(\zeta)\chi^\pm(\zeta)}{\zeta - z} d\zeta.$$

By definition we have again $g_a^+(z) + g_a^-(z) = g(z)$ when $z \in \mathring{D}$. So we finally define

$$S_a[g](z) := \sum_{n \geq 1} g_a^-(z - n) - \sum_{n \geq 0} g_a^+(z + n).$$

Lemma 4.15. *Let $h \in \mathcal{O}(D)$, $a \geq \frac{\pi}{4\beta}$ and $r = \max\{2\alpha, 2\beta\}$. Then $S_a : \mathcal{O}(D) \rightarrow \mathcal{O}(D)$ and*

$$\|S_a\| \leq C L (1 + a^{-1}) \log(r) \|\phi_a\|_D$$

for some $C > 0$ and $S_a[g]$ is a solution of equation (4.3).

Proof. It is trivial to check that formally $S_a[g]$ is a solution, so we only need to check that the sums converge and get the bound for the norm. For $z \in \mathring{D}$ and by the previous lemma we have

$$\begin{aligned} |S_a[g](z)| &\leq \left| \sum_{n \geq 1} g_a^-(z - n) \right| + \left| \sum_{n \geq 0} g_a^+(z + n) \right| \\ &\leq C L \log(r) \|\phi_a\|_D \|g\|_\infty \left(\left| \frac{1}{\phi_a(z)} \right| + \sum_{n \geq 1} \left| \frac{1}{\phi_a(z - n)} \right| + \sum_{n \geq 1} \left| \frac{1}{\phi_a(z + n)} \right| \right). \end{aligned}$$

Because $a \geq \frac{\pi}{4\beta}$, z stays far enough from the roots of ϕ_a so that $\phi_a(z)^{-1}$ stays bounded by $1/\sqrt{2}$. Then both sums can be bounded by some constant times the integral $\int_0^\infty e^{-as} ds$ and from this we get the result. \square

4.6.2 Approximation of fundamental solutions

The first difference equation is the one that the difference of the separatrices satisfies. We have

$$\begin{aligned} \delta(\varepsilon; \tau + 1) &= W^+(\varepsilon; \tau + 1) - W^-(\varepsilon; \tau + 1) \\ &= F_\varepsilon(W^+(\varepsilon; \tau)) - F_\varepsilon(W^-(\varepsilon; \tau)) \\ &= \left(\int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) ds \right) (W^+(\varepsilon; \tau) - W^-(\varepsilon; \tau)) \\ &= \left(\int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) ds \right) \delta(\varepsilon; \tau) \end{aligned}$$

so we write

$$\delta(\varepsilon; \tau + 1) = A(\varepsilon; \tau) \delta(\varepsilon; \tau)$$

with $A(\varepsilon; \tau) = \int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) ds$. We denote by $U(\varepsilon; \tau)$ the fundamental solution of this equation, i.e. a 2×2 matrix that satisfies

$$U(\varepsilon; \tau + 1) = A(\varepsilon; \tau) \cdot U(\varepsilon; \tau) \tag{4.4}$$

and $\det U(\varepsilon; \tau) = 1$.

For the second variational equation we define $D(\varepsilon; \tau) = F'_\varepsilon(W^-(\varepsilon; \tau))$ and we denote by $V(\varepsilon; \tau) = (\Xi(\varepsilon; \tau), (\dot{W}^-(\varepsilon; \tau)))$ a 2×2 matrix that satisfies

$$V(\varepsilon; \tau + 1) = D(\varepsilon; \tau) \cdot V(\varepsilon; \tau) \tag{4.5}$$

and $\det V(\varepsilon; \tau) = 1$.

The goal of this section is to prove that we can approximate U and V by the same function with errors that are of the same order. To this end we denote by R the 2×2 matrix which satisfies

$$A(\varepsilon; \tau) = D(\varepsilon; \tau) + R(\varepsilon; \tau)$$

and by Q the 2×2 matrix which satisfies

$$U(\varepsilon; \tau) = V(\varepsilon; \tau)(I + Q(\varepsilon; \tau)).$$

Here I denotes the identity matrix. Then we have

$$\begin{aligned} U(\varepsilon; \tau + 1) &= V(\varepsilon; \tau + 1)(I + Q(\varepsilon; \tau + 1)) \\ &= D(\varepsilon; \tau)V(\varepsilon; \tau)(I + Q(\varepsilon; \tau + 1)), \\ A(\varepsilon; \tau)U(\varepsilon; \tau) &= D(\varepsilon; \tau)U(\varepsilon; \tau) + R(\varepsilon; \tau)U(\varepsilon; \tau) \\ &= D(\varepsilon; \tau)V(\varepsilon; \tau)(I + Q(\varepsilon; \tau)) + R(\varepsilon; \tau)V(\varepsilon; \tau)(I + Q(\varepsilon; \tau)). \end{aligned}$$

From these we get the equation

$$Q(\varepsilon; \tau + 1) - Q(\varepsilon; \tau) = V^{-1}(\varepsilon; \tau) \cdot D^{-1}(\varepsilon; \tau) \cdot R(\varepsilon; \tau) \cdot V(\varepsilon; \tau)(I + Q(\varepsilon; \tau)). \quad (4.6)$$

Definition 4.16. We define the domains

$$\begin{aligned} \mathcal{M}_0 &:= \left\{ \tau \in \mathbb{C} : |\operatorname{Re}(\tau)| \leq 2, |\operatorname{Im}(\tau)| \leq \frac{\pi}{\varepsilon} - \frac{1}{\sqrt{\varepsilon}} \right\}, \\ \mathcal{M}^\pm &:= \left\{ \tau \in \mathbb{C} : |\operatorname{Re}(\tau)| \leq 2, \frac{\pi}{\varepsilon} - \frac{1}{\sqrt{\varepsilon}} \leq \pm \operatorname{Im}(\tau) \leq \frac{\pi}{\varepsilon} - \Lambda \right\}, \\ \mathcal{M} &:= \mathcal{M}_0 \cup \mathcal{M}^+ \cup \mathcal{M}^-. \end{aligned}$$

Definition 4.17. Let $M \in \mathbb{C}^\omega(\mathcal{M})^{2 \times 2}$. Then we define

$$\|M\|_{\sup} = \max_{i,j \in \{1,2\}} \sup_{t \in \mathcal{M}} |M_{ij}(t)|.$$

Lemma 4.18. Let $n > 8$ and let F_ε agree with the normal form up to order n . Then there exists $C_V > 0$ such that

$$\|V\|_{\sup} = \frac{C_V}{\varepsilon^4} \left(1 + O(\varepsilon^{1/2}) \right).$$

Proof. By writing $\Xi(\varepsilon; \tau) = (\xi_1(\varepsilon; \tau), \xi_2(\varepsilon; \tau))$ and $\dot{W}^-(\varepsilon; \tau) = (\zeta_1(\varepsilon; \tau), \zeta_2(\varepsilon; \tau))$ and eliminating from the equation as done in 3.21 we get the equation

$$\xi_2(\varepsilon; \tau + 1) = \left(D_{21}(\varepsilon; \tau) \frac{\zeta_1(\varepsilon; \tau)}{\zeta_2(\varepsilon; \tau)} + D_{22}(\varepsilon; \tau) \right) \xi_2(\varepsilon; \tau) + \frac{D_{21}(\varepsilon; \tau)}{\zeta_2(\varepsilon; \tau)}.$$

Evidently ζ_2 satisfies the homogeneous part of the above equation so we define

$\xi_2(\varepsilon; \tau) = C(\varepsilon; \tau) \zeta_2(\varepsilon; \tau)$ and by substitution we get

$$C(\varepsilon; \tau + 1) - C(\varepsilon; \tau) = \frac{D_{21}(\varepsilon; \tau)}{\zeta_2(\varepsilon; \tau + 1) \zeta_2(\varepsilon; \tau)} =: K(\varepsilon; \tau).$$

Combining the bounds we got for D_0 and D_1 , for all $\tau \in \mathcal{M}_0$ we have $W^-(\varepsilon; \tau) = \tilde{\mathcal{Z}}_n(\varepsilon; \tau) + O(\varepsilon^{\frac{n+1}{2}})$. We assume that $n > 8$ and we differentiate this relation to get

$$\begin{aligned} \zeta_2(\varepsilon; \tau) &= \frac{\varepsilon^2}{4b_{0,0}} \operatorname{sech}\left(\frac{\varepsilon\tau}{2}\right) + O(\varepsilon^3 \tanh\left(\frac{\varepsilon\tau}{2}\right)^3) \\ \zeta_1(\varepsilon; \tau) &= O(\varepsilon^3 \tanh\left(\frac{\varepsilon\tau}{2}\right)^3). \end{aligned}$$

The absolute value of $\operatorname{sech}(z)$ increases as $\operatorname{Im}(z)$ deviates from 0 and decreases as $\operatorname{Re}(z)$ deviates from 0. This implies that in \mathcal{M}_0 it is bounded from below by a constant independent of ε . We have

$$|\zeta_2(\varepsilon; \tau)^{-1}| = \frac{C'_0}{\varepsilon^2} (1 + O(\varepsilon))$$

for some $C'_0 > 0$. In order to bound $|\zeta_2(\varepsilon; \tau + 1)|$ from below we repeat the above process for $\tau \in \mathcal{M}_0 + 1$ and we see that the only thing that changes is the constant, i.e.

$$|\zeta_2(\varepsilon; \tau + 1)^{-1}| = \frac{C''_0}{\varepsilon^2} (1 + O(\varepsilon))$$

for some $C''_0 > 0$.

To get a bound for $D_{21}(\varepsilon; \tau)$, we recall that $[F'_\varepsilon(x, y)]_{21} = -2a_{0,1}\mu_1\varepsilon - 2b_{0,0}x + 2b_{0,0}^2xy + \dots$ so

$$|D_{21}(\varepsilon; \tau)| = C'''_0\varepsilon + O(\varepsilon^2 \tanh\left(\frac{\varepsilon\tau}{2}\right))$$

for some $C'''_0 > 0$ and since for any $\tau \in \mathcal{M}_0$ we have $\varepsilon \tanh\left(\frac{\varepsilon\tau}{2}\right) = O(\varepsilon^{1/2})$ we have

$$|K(\varepsilon; \tau)| = \frac{C_0}{\varepsilon^3} \left(1 + O(\varepsilon^{1/2})\right)$$

for some $C_0 > 0$.

When $\tau \in \mathcal{M}_1^+$ we need to use $\tilde{\mathfrak{W}}^-$ to get a bound. From the bound in D_2 we have $W^-(\varepsilon; \tau) = \tilde{\mathfrak{W}}_n^-(\varepsilon; \tau) + O(\varepsilon^{\frac{n-1}{2}})$, we assume that $n > 8$.

Recall that $\tau = t + \pi i/\varepsilon$, $\mathscr{W}_0^-(t) = (0, -(b_{0,0}t)^{-1}) + O(t^{-3})$ and $\mathscr{W}_n^-(t) = O(t^{n-1})$. Thus $\dot{\mathscr{W}}_0^-(t) = (0, b_{0,0}^{-1}t^{-2}) + O(t^{-4})$, $\dot{\mathscr{W}}_1^-(t) = O(t^{-2})$ and $\dot{\mathscr{W}}_n^-(t) = O(t^{n-2})$.

For $\tau \in \mathcal{M}_1^+$, $|t|$ is bounded from above by $\varepsilon^{-\frac{1}{2}}$ and from below by Λ . So in order to bound ζ_2 from below we need to estimate it for $\text{Im}(\tau) \approx \frac{\pi}{\varepsilon} - \frac{1}{\sqrt{\varepsilon}}$. In this region we get

$$\dot{\mathcal{W}}_0^-(t) = O(\varepsilon), \dot{\mathcal{W}}_1^-(t) = O(\varepsilon) \text{ and } \dot{\mathcal{W}}_n^-(t) = O(\varepsilon^{\frac{n}{2}-1}).$$

Using these we get

$$|\zeta_2(\varepsilon; \tau)^{-1}| = \frac{C'_1}{\varepsilon} \left(1 + O(\varepsilon^{1/2})\right)$$

for some $C'_1 > 0$. As above the same process on $\mathcal{M}_1^+ + 1$ gives the same bound with a different constant for $|\zeta_2(\varepsilon; \tau + 1)^{-1}|$.

Finally, on \mathcal{M}_1^+ we have $|D_{21}(\varepsilon; \tau)| = C''_1(1 + O(\varepsilon))$ so we get

$$|K(\varepsilon; \tau)| = \frac{C_1}{\varepsilon^2} \left(1 + O(\varepsilon^{1/2})\right)$$

for some $C_1 > 0$. Due to the real symmetry we get exactly the same bounds on \mathcal{M}_1^- .

Now that we know that K is bounded on \mathcal{M} we can use Lemma 4.15 to get the existence of C . We set $a = \varepsilon/2$, we have $r = 2\pi\varepsilon^{-1}$. Then

$$\|S_a\| \leq c' \varepsilon^{-2}$$

and

$$|C(\varepsilon; \tau)| \leq c'' \varepsilon^{-5}.$$

From this we get that

$$\xi_2(\varepsilon; \tau) = C(\varepsilon; \tau)\zeta_2(\varepsilon; \tau) = \frac{c_2}{\varepsilon^3} \left(1 + O(\varepsilon^{1/2})\right)$$

and

$$\xi_1(\varepsilon; \tau) = \frac{1}{\zeta_2(\varepsilon; \tau)} + \frac{\xi_2(\varepsilon; \tau)\zeta_1(\varepsilon; \tau)}{\zeta_2(\varepsilon; \tau)} = \frac{c_1}{\varepsilon^4} \left(1 + O(\varepsilon^{1/2})\right).$$

Getting the maximum of these bounds gives the result. \square

Lemma 4.19. *Let $n \geq 20$ and let F_ε agree with the normal form up to order n . Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ there exists a constant $C_Q > 0$ such that $\|Q\|_\infty \leq C_Q \varepsilon^{\frac{n-19}{2}} (1 + O(\varepsilon^{1/2}))$.*

Proof. We know that $\|D^{-1}\|_{\text{sup}} = 1 + O(\varepsilon^{1/2})$ and since $\det V = 1$, we have $\|V^{-1}\|_{\text{sup}} = C_V \varepsilon^{-4} (1 + O(\varepsilon^{1/2}))$. We define

$$M = V^{-1} \cdot D^{-1} \cdot R \cdot V.$$

Then equation (4.6) becomes

$$Q(\tau + 1) - Q(\tau) = M(\tau) + M(\tau) \cdot Q(\tau).$$

From this we get

$$Q(\tau) = S_a[M](\tau) + S_a[M \cdot Q](\tau).$$

We define

$$\mathbf{X} : Q \mapsto S_a[M] + S_a[M \cdot Q].$$

If W^+ and W^- coincide with the normal form up to order n , then there exists $C_n > 0$ such that $\|R\|_{\text{sup}} = C_n \varepsilon^{\frac{n+1}{2}} (1 + O(\varepsilon^{1/2}))$. Then there exists $C_M > 0$ such that $\|M\|_{\text{sup}} = C_M \varepsilon^{\frac{n-15}{2}} (1 + O(\varepsilon^{1/2}))$. Recall that $\|S_a\| \leq c' \varepsilon^{-2}$, so

$$\|S_a[M]\|_{\infty} \leq C'_M \varepsilon^{\frac{n-19}{2}} (1 + O(\varepsilon^{1/2}))$$

and of course

$$\|S_a[M \cdot Q]\|_{\infty} \leq C'_M \varepsilon^{\frac{n-19}{2}} (1 + O(\varepsilon^{1/2})) \|Q\|_{\infty}.$$

Then for $n \geq 20$ the operator \mathbf{X} is a contraction on some neighbourhood of the origin $\mathcal{V}_c = \{x \in \mathbb{C}^\omega(\mathcal{M})^{2 \times 2} : \|x\|_{\infty} \leq c \|S_a[M]\|_{\infty}\}$ for big enough c and small enough ε . This means that Q is the fixed point of \mathbf{X} and from this the result follows. \square

Corollary 4.20. $U = V + O(\varepsilon^{\frac{n-27}{2}})$.

4.7 Sharper bounds

4.7.1 Upper bound for the splitting

With everything that is known up to this point we can prove that the splitting has an exponentially small upper bound.

Lemma 4.21. *For all $\tau \in [-2, 2]$ there exists a constant $C > 0$ such that*

$$|\delta(\varepsilon; \tau)| \leq C\varepsilon^{-2}e^{-\frac{2\pi^2}{\varepsilon}}.$$

Before we prove this lemma we need some results on real analytic periodic functions.

Real analytic periodic functions in a rectangular domain

Lemma 4.22. *Let $D = \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq \alpha, |\operatorname{Im}(z)| \leq \beta\}$ for some $\alpha, \beta \geq 1$ and let g be a real analytic function in D and continuous on ∂D such that $g(\tau+1) = g(\tau)$ when both τ and $\tau+1$ are in D . Moreover, we assume that there exists $\tau_h \in [-\alpha, \alpha]$ such that $g(\tau_h) = 0$. We write g as a Fourier series:*

$$g(\tau) = g_0 + \sum_{n \geq 1} g_n e^{-2\pi n i \tau} + \sum_{n \geq 1} \overline{g_n} e^{2\pi n i \tau},$$

for some $g_n \in \mathbb{C}$. Then it is true that

$$|g_n| \leq \|g\|_{\infty} e^{-2\pi \beta n}$$

for all $n \in \mathbb{N}$ and

$$|g_0| \leq 4\|g\|_{\infty} e^{-2\pi \beta}.$$

Proof. By setting $\tau = i\beta$ we get

$$g(i\beta) = g_0 + \sum_{n \geq 1} g_n e^{2\pi \beta n} + \sum_{n \geq 1} \overline{g_n} e^{-2\pi \beta n}$$

and this implies that

$$|g_n| \leq \|g\|_{\infty} e^{-2\pi \beta n}$$

for all $n \in \mathbb{N}_0$.

From the equation $g(\tau_h) = 0$ we get

$$|g_0| \leq 2 \sum_{n \geq 1} |g_n|.$$

This sum is a geometric progression and we get

$$|g_0| \leq 2\|g\|_{\infty} e^{-2\pi \beta} \frac{1}{1 - e^{-2\pi \beta}} \leq 4\|g\|_{\infty} e^{-2\pi \beta}. \quad \square$$

Corollary 4.23. *Let g and D be as described above. Then for all $\tau \in [-\alpha, \alpha]$ it is true that*

$$|g(\tau)| \leq 8\|g\|_{\infty} e^{-2\pi\beta}.$$

Bound for δ

Let $U(\varepsilon; \tau) = (\Psi(\varepsilon; \tau), \Phi(\varepsilon; \tau))$. Then there exist two functions $\Theta(\varepsilon; \tau)$ and $q(\varepsilon; \tau)$ such that

$$\delta(\varepsilon; \tau) = \Theta(\varepsilon; \tau) \Psi(\varepsilon; \tau) + q(\varepsilon; \tau) \Phi(\varepsilon; \tau).$$

Then evidently we have

$$\Theta(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \Phi(\varepsilon; \tau))$$

and

$$\begin{aligned} \Theta(\varepsilon; \tau + 1) &= \omega(\delta(\varepsilon; \tau + 1), \Phi(\varepsilon; \tau + 1)) \\ &= \omega(A(\varepsilon; \tau) \delta(\varepsilon; \tau), A(\varepsilon; \tau) \Phi(\varepsilon; \tau)) \\ &= \omega(\delta(\varepsilon; \tau), \Phi(\varepsilon; \tau)) \\ &= \Theta(\varepsilon; \tau). \end{aligned}$$

Similarly we get that $q(\varepsilon; \tau + 1) = q(\varepsilon; \tau)$.

Because the map is area-preserving, there has to be a homoclinic point $W^-(\varepsilon; \tau_h)$ such that $\delta(\varepsilon; \tau_h) = 0$. Because Ψ and Φ are linearly independent this implies that $\Theta(\varepsilon; \tau_h) = q(\varepsilon; \tau_h) = 0$.

Both Θ and q are defined in a rectangular domain with $\alpha = 2$ and $\beta = \frac{\pi}{\varepsilon} - \Lambda$. Now we apply Corollary 4.23 and we get that there exists a constant $C > 0$ such that for all $\tau \in [-2, 2]$ it holds that

$$|\Theta(\varepsilon; \tau)|, |q(\varepsilon; \tau)| \leq C e^{-\frac{2\pi^2}{\varepsilon}}.$$

To prove Lemma 4.21 we just combine those bounds with the bounds of Ψ and Φ .

4.7.2 Variational equations revisited

In order to prove Lemma 4.19 we used the fact that the stable and unstable solutions can be approximated by the same formal series. This gives an error that is polynomially small with ε . However, we saw in the previous section that the splitting is actually exponentially small. We can now use this result to get a sharper bound on the difference of the two fundamental solutions.

Lemma 4.24. *Let $D = \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq 2, |\operatorname{Im}(z)| \leq 1\}$. Then there exists $C > 0$ such that on D it is true that*

$$\|U - V\|_{\sup} \leq C\varepsilon^{-16} e^{-\frac{2\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})).$$

Proof. The proof is essentially the same as the proof of Lemma 4.19. Here we restate the main points.

By definition we have

$$\begin{aligned} A(\varepsilon; \tau) &= \int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) \, ds \\ &= \int_0^1 F'_\varepsilon(W^-(\varepsilon; \tau) + s \delta(\varepsilon; \tau)) \, ds. \end{aligned}$$

Then

$$\begin{aligned} R(\varepsilon; \tau) &= A(\varepsilon; \tau) - D(\varepsilon; \tau) \\ &= \int_0^1 \left(F'_\varepsilon(W^-(\varepsilon; \tau) + s \delta(\varepsilon; \tau)) - F'_\varepsilon(W^-(\varepsilon; \tau)) \right) \, ds \end{aligned}$$

and by using Taylor's theorem and the bound for δ we get that there exists $C > 0$ such that for all $\tau \in [-2, 2]$ it holds that

$$|R(\varepsilon; \tau)| \leq C\varepsilon^{-2} e^{-\frac{2\pi^2}{\varepsilon}}.$$

This bound can extend to D by increasing the constant.

We have

$$M = V^{-1} \cdot D^{-1} \cdot R \cdot V$$

and by assuming that M is a function defined on D we get

$$\begin{aligned}\|V\|_{\text{sup}}, \|V^{-1}\|_{\text{sup}} &\leq C \varepsilon^{-4} (1 + O(\varepsilon^{1/2})) \\ \|D^{-1}\|_{\text{sup}} &\leq 1 + O(\varepsilon^{1/2}) \\ \|S_a\|_{\text{sup}} &\leq Ch \varepsilon^{-2}.\end{aligned}$$

Recall that we set $a = \varepsilon/2$. Then

$$\|S_a[M]\|_{\text{sup}} \leq C \varepsilon^{-12} e^{-\frac{2\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})).$$

Now by the same contraction mapping argument we get

$$\|Q\|_{\text{sup}} \leq C \varepsilon^{-12} e^{-\frac{2\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})),$$

which implies

$$\|U - V\|_{\text{sup}} = \|V \cdot Q\|_{\text{sup}} \leq C \varepsilon^{-16} e^{-\frac{2\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})). \quad \square$$

4.8 Asymptotic expansion of the separatrix splitting

We have defined

$$\Theta(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \Phi(\varepsilon; \tau))$$

and we also define

$$\Theta^-(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau)).$$

Unlike Θ , Θ^- is not periodic.

We write Θ as a Fourier series

$$\Theta(\varepsilon; \tau) = c_0 + \sum_{n \geq 1} c_n(\varepsilon) e^{-2\pi n i \tau} + \sum_{n \geq 1} \overline{c_n(\varepsilon)} e^{2\pi n i \tau}$$

and from this we get

$$\Theta(\varepsilon; t + \frac{\pi}{\varepsilon} i) = c_0 + \sum_{n \geq 1} c_n(\varepsilon) e^{\frac{2\pi^2 n}{\varepsilon}} e^{-2\pi n i t} + \sum_{n \geq 1} \overline{c_n(\varepsilon)} e^{-\frac{2\pi^2 n}{\varepsilon}} e^{2\pi n i t}.$$

Then we define

$$\theta(\varepsilon) := \int_{L_1(\nu)} e^{2\pi i t} \Theta(\varepsilon; t + \frac{\pi}{\varepsilon} i) dt = c_1(\varepsilon) e^{\frac{2\pi^2}{\varepsilon}}$$

with $\nu > 0$ and $L_1(\nu) = \{t \in \mathbb{C} : \text{Im}(t) = -\nu \text{ and } |\text{Re}(t)| \leq \frac{1}{2}\}$. As usual $t = \tau - \frac{\pi}{\varepsilon}\mathbf{i}$ with $\text{Im}(t) < 0$. Here we fix $\nu = -(M+2)(2\pi)^{-1}\log(\varepsilon)$. This implies that $e^{2\pi\mathbf{i}t} = O(\varepsilon^{-M-2})$.

We know that for $t \in L_1(\nu)$, Φ can be approximated by $\dot{\mathfrak{W}}^-$, i.e.⁶

$$\Phi(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) = \dot{\mathfrak{W}}_N^-(\varepsilon; t) + O(\varepsilon^{2M+4}) = \sum_{n=0}^N \varepsilon^n \dot{\mathcal{W}}_n^-(t) + O(\varepsilon^{2M+4})$$

and that

$$\dot{W}^-(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) = \dot{\mathfrak{W}}_N^-(\varepsilon; t) + O(\varepsilon^{2M+17}).$$

From these we get

$$\Phi(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) = \dot{W}^-(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) + O(\varepsilon^{2M+4})$$

and

$$\Theta^-(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) = \Theta(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) + O(\varepsilon^{2M+3}).$$

4.8.1 The constant term of Θ

Lemma 4.25. *There exists $C > 0$ such that*

$$|c_0| \leq C\varepsilon^{-18}e^{-\frac{4\pi^2}{\varepsilon}}(1 + O(\varepsilon^{1/2})).$$

Proof. Let τ_h be such that $W^+(\varepsilon; \tau_h) = W^-(\varepsilon; \tau_h)$ and of course $W^+(\varepsilon; \tau_h + 1) = W^-(\varepsilon; \tau_h + 1)$. By definition we have that

$$c_0 = \int_0^1 \omega(\delta(\varepsilon; \tau_h + s), \Phi(\varepsilon; \tau_h + s)) ds.$$

We define

$$\sigma_0 = \int_0^1 \omega(\delta(\varepsilon; \tau_h + s), \dot{W}^-(\varepsilon; \tau_h + s)) ds.$$

Then

$$|c_0 - \sigma_0| \leq \int_0^1 |\omega(\delta(\varepsilon; \tau_h + s), \Phi(\varepsilon; \tau_h + s) - \dot{W}^-(\varepsilon; \tau_h + s))| ds.$$

Using lemma 4.21 we get that

$$|c_0 - \sigma_0| \leq C\varepsilon^{-18}e^{-\frac{4\pi^2}{\varepsilon}}(1 + O(\varepsilon^{1/2})).$$

⁶ Recall that $N = 4M + 35$.

Let A be the signed area enclosed by these two pieces of the separatrices. Using Green's formula to calculate the area we get that

$$A = \frac{1}{2} \int_0^1 \omega(W^+(\varepsilon; \tau_h + s), \dot{W}^+(\varepsilon; \tau_h + s)) - \omega(W^-(\varepsilon; \tau_h + s), \dot{W}^-(\varepsilon; \tau_h + s)) ds.$$

It holds that $W^+ = W^- + \delta$ so we have

$$\omega(W^+, \dot{W}^+) - \omega(W^-, \dot{W}^-) = \omega(\delta, \dot{W}^-) + \omega(W^-, \dot{\delta}) + \omega(\delta, \dot{\delta})$$

and from these

$$\begin{aligned} \omega(\delta, \dot{W}^-) - \frac{1}{2} \left(\omega(W^+, \dot{W}^+) - \omega(W^-, \dot{W}^-) \right) &= \frac{1}{2} \left(\omega(\delta, \dot{W}^-) - \omega(W^-, \dot{\delta}) - \omega(\delta, \dot{\delta}) \right) \\ &= \frac{1}{2} \left(\frac{d}{dt} \omega(\delta, W^-) - \omega(\delta, \dot{\delta}) \right). \end{aligned}$$

Combining the above we get

$$\sigma_0 - A = \omega(\delta(\varepsilon; \tau_h + s), W^-(\varepsilon; \tau_h + s)) \Big|_{s=0}^1 - \int_0^1 \frac{1}{2} \omega(\delta(\varepsilon; \tau_h + s), \dot{\delta}(\varepsilon; \tau_h + s)) ds.$$

We know that the δ and $\dot{\delta}$ are bounded by $C\varepsilon^{-2} e^{-\frac{2\pi^2}{\varepsilon}}$ and from the area-preservation of the map we get that $A = 0$. This leads to the result. \square

4.8.2 The first Fourier coefficient of Θ

Lemma 4.26. *There exist constants $\theta_i \in \mathbb{C}$ such that*

$$\theta(\varepsilon) = \sum_{n=0}^M \varepsilon^n \theta_n + O(\varepsilon^{M+1}).$$

To prove this lemma we need to also approximate Θ using the formal solution. We already know that for $t \in L(\nu)$ we have

$$W^\pm(\varepsilon; t + \frac{\pi}{\varepsilon} \mathfrak{i}) = \tilde{\mathfrak{W}}_N^\pm(\varepsilon; t) + O(\varepsilon^{2M+17}) = \sum_{n=0}^N \varepsilon^n \mathscr{W}_n^\pm(t) + O(\varepsilon^{2M+17}).$$

We define the formal series

$$\tilde{\delta}(\varepsilon; t) = \sum_{n \geq 0} \varepsilon^n \delta_n(t)$$

with $\delta_n(t) = \mathscr{W}_n^+(t) - \mathscr{W}_n^-(t)$ and we denote by $\bar{\delta}_N$ its truncation to order N . So we write

$$\delta(\varepsilon; t + \frac{\pi}{\varepsilon} \mathbf{i}) = \bar{\delta}_N(\varepsilon; t) + O(\varepsilon^{2M+17}) = \sum_{n=0}^N \varepsilon^n \delta_n(t) + O(\varepsilon^{2M+17}).$$

Recall that $\delta_n(t) = O(t^{n+2}e^{-2\pi \mathbf{i}t})$. Then we have

$$\begin{aligned} \omega(\delta(\varepsilon; t + \frac{\pi}{\varepsilon} \mathbf{i}), \Phi(\varepsilon; t + \frac{\pi}{\varepsilon} \mathbf{i})) &= \\ &= \omega\left(\bar{\delta}_N(\varepsilon; t)(\varepsilon; t) + O(\varepsilon^{2M+15}), \dot{\mathfrak{W}}_N^-(\varepsilon; t) + O(\varepsilon^{2M+4})\right) \\ &= \omega\left(\bar{\delta}_N(\varepsilon; t)(\varepsilon; t), \dot{\mathfrak{W}}_N^-(\varepsilon; t)\right) + \omega\left(\bar{\delta}_N(\varepsilon; t)(\varepsilon; t), O(\varepsilon^{2M+4})\right) \\ &\quad + \omega\left(O(\varepsilon^{2M+17}), \dot{\mathfrak{W}}_N^-(\varepsilon; t)\right) + \omega\left(O(\varepsilon^{2M+17}), O(\varepsilon^{2M+4})\right) \\ &= \omega\left(\bar{\delta}_N(\varepsilon; t)(\varepsilon; t), \dot{\mathfrak{W}}_N^-(\varepsilon; t)\right) + O(\varepsilon^{2M+3}). \end{aligned}$$

Asymptotic series for Θ

Let

$$\begin{aligned} \tilde{\Theta}(\varepsilon; t) &= \omega\left(\tilde{\delta}(\varepsilon; t), \dot{\mathfrak{W}}^-(\varepsilon; t)\right), \\ \tilde{\Theta}(\varepsilon; t) &= \sum_{n \geq 0} \varepsilon^n \zeta_n(t) \end{aligned}$$

and

$$\tilde{\Theta}_N(\varepsilon; t) = \sum_{n=0}^N \varepsilon^n \zeta_n(t).$$

By the previous section we can approximate Θ by $\tilde{\Theta}$ and we get

$$\Theta(\varepsilon; t + \frac{\pi}{\varepsilon} \mathbf{i}) = \tilde{\Theta}_N(\varepsilon; t) + O(\varepsilon^{2M+1})$$

and from this we get

$$\begin{aligned} \theta(\varepsilon) &= \int_{L_1(\nu)} e^{2\pi \mathbf{i}t} \left(\tilde{\Theta}_N(\varepsilon; t) + O(\varepsilon^{2M+1}) \right) dt \\ &= \sum_{n=0}^N \varepsilon^n \left(\int_{L_1(\nu)} e^{2\pi \mathbf{i}t} \zeta_n(t) dt \right) + O(\varepsilon^{M+1}). \end{aligned}$$

By the definition of $\tilde{\Theta}$ we get

$$\tilde{\Theta}(\varepsilon; t + 1) = \omega\left(\tilde{\delta}(\varepsilon; t + 1), \dot{\mathfrak{W}}^-(\varepsilon; t + 1)\right)$$

$$\begin{aligned}
&= \omega \left(\tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^+(\varepsilon; t)) - \tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)), \tilde{F}'_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) \cdot \dot{\tilde{\mathfrak{W}}}^-(\varepsilon; t) \right) \\
&= \tilde{\Theta}(\varepsilon; t) + \omega \left(\tilde{\mathfrak{V}}(\varepsilon; t), \tilde{F}'_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) \cdot \dot{\tilde{\mathfrak{W}}}^-(\varepsilon; t) \right) \\
&= \tilde{\Theta}(\varepsilon; t) + \omega \left(\tilde{\mathfrak{V}}(\varepsilon; t), \dot{\tilde{\mathfrak{W}}}^-(\varepsilon; t+1) \right)
\end{aligned}$$

with

$$\tilde{\mathfrak{V}}(\varepsilon; t) = \tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t) + \tilde{\delta}(\varepsilon; t)) - \tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) - \tilde{F}'_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) \cdot \tilde{\delta}(\varepsilon; t).$$

We will show that $\tilde{\mathfrak{V}}$ can be written as

$$\tilde{\mathfrak{V}}(\varepsilon; t) = \sum_{n \geq 0} \varepsilon^n \mathcal{V}_n(t)$$

with $\mathcal{V}_n(t) = O(t^{n+4}e^{-4\pi i t})$.

For an analytic $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ we write its Taylor series as

$$H(W + v) = \sum_{n \geq 0} \frac{1}{n!} H^{(n)}(W; \underbrace{v, \dots, v}_{n\text{-times}}),$$

where $H^{(n)}$ has to be viewed as a symmetric tensor. Notice that the tensor is not linear with respect to its first argument.

Using this notation we write $H'(W) \cdot u$ as $H^{(1)}(W; u)$ and we have

$$H^{(1)}(W + v; u) = \sum_{n \geq 0} \frac{1}{n!} H^{(n+1)}(W; \underbrace{v, \dots, v, u}_{n\text{-times}}).$$

Finally, it holds

$$H^{(n)}(W + u, \underbrace{v, \dots, v}_{n\text{-times}}) = H^{(n)}(W, \underbrace{v, \dots, v}_{n\text{-times}}) + H^{(n+1)}(W, \underbrace{v, \dots, v, u}_{n\text{-times}}) + O(v^n u^2).$$

We also need a slight generalization of the multi-index notation. We define the set

$$\mathcal{P}(n, m) := \left\{ (k_1, \dots, k_m) \in \mathbb{N}^k : \sum_{i=1}^m k_i = n \right\},$$

which is the of all m -tuples of positive integers whose sum is n . Then we define the

set

$$\hat{\mathcal{P}}(n, m) := \left\{ (k_1, \dots, \hat{k}_j, \dots, k_m) \in \mathbb{N}^m : \sum_{i=1}^m k_i = n \right\},$$

which is the same as the previous one with the exception that there is a distinguished integer that we mark out. Finally we define the set

$$\hat{\mathcal{P}}_o(n, m) := \left\{ (k_1, \dots, k_{m-1}, \hat{k}_m) \in \mathbb{N}^m : \sum_{i=1}^{m+1} k_i = n \right\},$$

which it a subset of the above, since the distinguished integer is always at the last place.

Using these we define

$$\begin{aligned} \overline{\mathcal{W}}_{(k_1, \dots, k_m)} &= (\mathcal{W}_{k_1}^-, \dots, \mathcal{W}_{k_m}^-), \\ \overline{\mathcal{W}}_{(k_1, \dots, \hat{k}_j, \dots, k_m)} &= (\mathcal{W}_{k_1}^-, \dots, \delta_{k_j}, \dots, \mathcal{W}_{k_m}^-), \\ \overline{\mathcal{W}}_{(k_1, \dots, k_{m-1}, \hat{k}_m)} &= (\mathcal{W}_{k_1}^-, \dots, \mathcal{W}_{k_{m-1}}^-, \delta_{k_m}). \end{aligned}$$

Using this notation and having in mind that for any bounded linear map A , it holds $A(\delta_i(t), \delta_j(t)) = O(t^k e^{-4\pi i t})$ with $k = i + j + 4$, we expand in Taylor series keeping only terms that are independent or linear in any δ_i and we get

$$\begin{aligned} \tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^- + \tilde{\delta}) &= \sum_{n \geq 0} \varepsilon^n \left(\tilde{F}_n(\mathcal{W}_0^- + \delta_0) + \sum_{m=0}^{n-1} \sum_{k=1}^{n-m} \sum_{p \in \mathcal{P}(n-m, k)} \frac{1}{k!} \tilde{F}_m^{(k)}(\mathcal{W}_0^- + \delta_0; \overline{\mathcal{W}}_p) \right. \\ &\quad \left. + \sum_{m=0}^{n-1} \sum_{k=1}^{n-m} \sum_{p \in \hat{\mathcal{P}}(n-m, k)} \frac{1}{k!} \tilde{F}_m^{(k)}(\mathcal{W}_0^- + \delta_0; \overline{\mathcal{W}}_p) + O(t^{n+4} e^{-4\pi i t}) \right), \end{aligned}$$

$$\begin{aligned} \tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) &= \sum_{n \geq 0} \varepsilon^n \left(\tilde{F}_n(\mathcal{W}_0^-) \right. \\ &\quad \left. + \sum_{m=0}^{n-1} \sum_{k=1}^{n-m} \sum_{p \in \mathcal{P}(n-m, k)} \frac{1}{k!} \tilde{F}_m^{(k)}(\mathcal{W}_0^-; \overline{\mathcal{W}}_p) + O(t^{n+4} e^{-4\pi i t}) \right), \end{aligned}$$

and

$$\begin{aligned} \tilde{F}'_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) \cdot \delta(\varepsilon; t) &= \sum_{n \geq 0} \varepsilon^n \left(\sum_{m=0}^n \tilde{F}_m^{(1)}(\mathcal{W}_0^-; \delta_{n-m}) \right. \\ &\quad \left. + \sum_{m=0}^{n-1} \sum_{k=1}^{n-m} \sum_{p \in \mathcal{P}(n-m, k)} \frac{1}{k!} \tilde{F}_m^{(k+1)}(\mathcal{W}_0^-; \overline{\mathcal{W}}_p, \delta_0) \right) \end{aligned}$$

$$+ \sum_{m=0}^{n-1} \sum_{k=2}^{n-m} \sum_{p \in \hat{\mathcal{P}}_o(n-m,k)} \frac{1}{(k-1)!} \tilde{F}_m^{(k)}(\mathcal{W}_0^-; \overline{\mathcal{W}}_p) + O(t^{n+4} e^{-4\pi i t}) \Bigg).$$

We fix n, m, k and $p \in \mathcal{P}(n-m, k)$ and by expanding in Taylor series the first term we see that

$$\frac{1}{k!} \tilde{F}_m^{(k)}(\mathcal{W}_0^- + \delta_0; \overline{\mathcal{W}}_p) - \frac{1}{k!} \tilde{F}_m^{(k)}(\mathcal{W}_0^-; \overline{\mathcal{W}}_p) - \frac{1}{k!} \tilde{F}_m^{(k+1)}(\mathcal{W}_0^-; \overline{\mathcal{W}}_p, \delta_0)$$

is of order δ_0^2 .

Then we fix n, m , set $k = 1$ and we see that

$$\tilde{F}_m^{(1)}(\mathcal{W}_0^- + \delta_0; \delta_{n-m}) - \tilde{F}_m^{(1)}(\mathcal{W}_0^-; \delta_{n-m})$$

is of order $\delta_0 \delta_{n-m}$.

Finally we fix $n, m, k > 1$ and we see that

$$\begin{aligned} & \sum_{p \in \hat{\mathcal{P}}(n-m,k)} \frac{1}{k!} \tilde{F}_m^{(k)}(\mathcal{W}_0^- + \delta_0; \overline{\mathcal{W}}_p) - \sum_{p \in \hat{\mathcal{P}}_o(n-m,k)} \frac{1}{(k-1)!} \tilde{F}_m^{(k)}(\mathcal{W}_0^-; \overline{\mathcal{W}}_p) \\ &= \sum_{p \in \hat{\mathcal{P}}_o(n-m,k)} \frac{1}{(k-1)!} \tilde{F}_m^{(k)}(\mathcal{W}_0^- + \delta_0; \overline{\mathcal{W}}_p) - \frac{1}{(k-1)!} \tilde{F}_m^{(k)}(\mathcal{W}_0^-; \overline{\mathcal{W}}_p). \end{aligned}$$

So we see each term of the sum is of order $\delta_0 \delta_j$ for some $j \in \{1, \dots, n-m\}$.

The above show that for all n , $\mathcal{V}_n(t) = O(t^{n+4} e^{-4\pi i t})$.

Asymptotic series for $\theta(\varepsilon)$

Recall that it holds

$$\theta(\varepsilon) = \sum_{n=0}^N \varepsilon^n \left(\int_{L_1(\nu)} e^{2\pi i t} \zeta_n(t) dt \right) + O(\varepsilon^{M+1}). \quad (4.7)$$

We define

$$L_1^-(\mu) := \bigcup_{\kappa \geq \mu} L_1(\kappa)$$

and

$$L^-(\mu) := \bigcup_{\kappa \geq \mu} \left(L_1(\kappa) \cup (L_1^-(\kappa) + 1) \right).$$

From the previous section we get that

$$\zeta_n(t+1) = \zeta_n(t) + \sum_{m=0}^n \omega(\mathcal{V}_m(t), \dot{\mathcal{W}}_{n-m}^-(t+1))$$

and this implies that

$$e^{2\pi i(t+1)} \zeta_n(t+1) = e^{2\pi i t} \zeta_n(t) + r_n(t)$$

with

$$r(t) = \sum_{m=0}^n \omega(e^{2\pi i t} \mathcal{V}_m(t), \dot{\mathcal{W}}_{n-m}^-(t+1))$$

and $r_n(t) = O(t^{n+2}e^{-2\pi i t})$. All of the above functions are analytic in $L^-(\nu)$.

For all $t \in L_1^-(\nu)$ we define

$$\rho_n(t) = \int_t^{t+1} e^{2\pi i s} \zeta_n(s) ds.$$

This satisfies the equation

$$\rho_n(t+1) = \rho_n(t) + \int_t^{t+1} r_n(s) ds,$$

which has as solution

$$\rho_n(t) = \theta_n + \int_{-\mathrm{i}\infty}^t r_n(s) ds$$

for some constant θ_n .

Since we know the bound for r_n and it holds

$$\begin{aligned} \int_{-\infty}^{|t|} s^{n+2} e^{-2\pi s} ds &= (2\pi)^{-n-3} \Gamma(n+3, 2\pi|t|) \\ &\leq C_\nu |t|^{n+2} e^{-2\pi|t|} \end{aligned}$$

we get that for all $t \in L_1^-(\nu)$

$$\rho_n(t) = \theta_n + O(t^{n+2}e^{-2\pi i t}).$$

As ν was chosen such that $e^{-2\pi i t} = O(\varepsilon^{M+2})$, then for any $n \in \mathbb{N}$ it holds $t^{n+2}e^{-2\pi i t} = O(\varepsilon^{M+1})$. This gives

$$\int_{L_1(\nu)} e^{2\pi i t} \zeta_n(t) dt = \theta_n + O(\varepsilon^{M+1}),$$

which we can combine with the equation (4.7) to get

$$\theta(\varepsilon) = \sum_{n=0}^M \varepsilon^n \theta_n + O(\varepsilon^{M+1}).$$

Remark. For the first constant θ_0 we have

$$\theta_0 = \int_{L_1(\nu)} e^{2\pi i t} \omega(\delta_0(t), \dot{\mathcal{W}}_0^-(t)) dt,$$

which is approximately the Stokes constant of the resonant map.

4.8.3 Asymptotic series for the homoclinic invariant

Now we have all the ingredients we need to prove the asymptotic for the Lazutkin homoclinic invariant.

Lemma 4.27. *There exist real numbers ϑ_n such that*

$$\Omega(\varepsilon) = \left(\sum_{n=0}^M \vartheta_n \varepsilon^n + O(\varepsilon^{M+1}) \right) e^{-\frac{2\pi^2}{\varepsilon}}.$$

Moreover $\vartheta_0 = 4\pi|\theta_0|$, where θ_0 is the Stokes constant of the resonant map.

Proof. By lemma 4.24 we have

$$\Phi(\varepsilon; \tau) - \dot{W}^-(\varepsilon; \tau) = O(\varepsilon^{-16} e^{-\frac{2\pi^2}{\varepsilon}}).$$

This implies that since

$$\Theta(\varepsilon; \tau) - \Theta^-(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \Phi(\varepsilon; \tau) - \dot{W}^-(\varepsilon; \tau)),$$

using the improved bound of lemma 4.21 for real τ we get

$$\Theta(\varepsilon; \tau) - \Theta^-(\varepsilon; \tau) = O(\varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}})$$

and we know that

$$\Theta(\varepsilon; \tau) = c_0 + \theta(\varepsilon) e^{-\frac{2\pi^2}{\varepsilon}} e^{-2\pi i \tau} + \overline{\theta(\varepsilon)} e^{-\frac{2\pi^2}{\varepsilon}} e^{2\pi i \tau} + O(e^{-\frac{4\pi^2}{\varepsilon}}).$$

Since we know that $|c_0| \leq C \varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2}))$, we get

$$\Theta^-(\varepsilon; \tau) = 2|\theta(\varepsilon)| e^{-\frac{2\pi^2}{\varepsilon}} \cos(2\pi\tau - \arg(\theta(\varepsilon))) + O(\varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}}).$$

Let $W^+(\tau_h)$ be a homoclinic point. Then evidently $\Theta^-(\varepsilon; \tau_h) = 0$ and from the above relation we get that

$$2\pi\tau_h - \arg(\theta(\varepsilon)) = 2\pi k + O(\varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}})$$

for some $k \in \mathbb{Z}$. This implies that

$$\dot{\Theta}^-(\varepsilon; \tau_h) = 4\pi|\theta(\varepsilon)| e^{-\frac{2\pi^2}{\varepsilon}} + O(\varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}}).$$

Differentiating $\Theta^-(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau))$ we get

$$\dot{\Theta}^-(\varepsilon; \tau) = \omega(\dot{\delta}(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau)) + \omega(\delta(\varepsilon; \tau), \ddot{W}^-(\varepsilon; \tau))$$

Since $\delta(\varepsilon; \tau_h) = 0$ we get

$$\begin{aligned} \dot{\Theta}^-(\varepsilon; \tau_h) &= \omega(\dot{\delta}(\varepsilon; \tau_h), \dot{W}^-(\varepsilon; \tau_h)) \\ &= \omega(\dot{W}^+(\varepsilon; \tau_h), \dot{W}^-(\varepsilon; \tau_h)), \end{aligned}$$

which is by definition the homoclinic invariant.

Finally to prove the lemma we use the fact that $\theta(\varepsilon) = \sum_{n=0}^M \varepsilon^n \theta_n + O(\varepsilon^{M+1})$. This implies that

$$4\pi|\theta(\varepsilon)| = \sum_{n=0}^M \vartheta_n \varepsilon^n + O(\varepsilon^{M+1})$$

for some real constants ϑ_n . □

Chapter 5

Computation of the Stokes constant

The goal of this chapter is to give a procedure to calculate the Stokes constant. Recall that

$$\theta = \lim_{t \rightarrow +\infty} e^{2\pi t} \omega(W^+(-it) - W^-(-it), \dot{W}^-(-it)).$$

The convergence in this case is exponential. In other words, for $t \in \mathbb{R}^+$ we define

$$\theta_{\text{app}}(t) = e^{2\pi t} \omega(W^+(-it) - W^-(-it), \dot{W}^-(-it))$$

and we know that the difference $\theta - \theta_{\text{app}}(t)$ is of order $t^5 e^{-2\pi t}$.

We will see that θ can be approximated numerically with known rate. However, a method for a computer assisted proof is not provided here.

5.1 Approximation of the separatrices for a map close to the normal form

Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a map at 1:3 resonance that agrees with the normal form for the first three orders and let $F = f^3$. Recall that

$$F(x, y) = \begin{pmatrix} x - 2b_0xy + b_0^2(x^3 + xy^2) \\ y - b_0(x^2 - y^2) + b_0^2(x^2y + y^3) \end{pmatrix} + R_f(x, y)$$

with $R_f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that $\|R_f(x, y)\|_\infty \leq c_f \|(x, y)\|_\infty^4$.

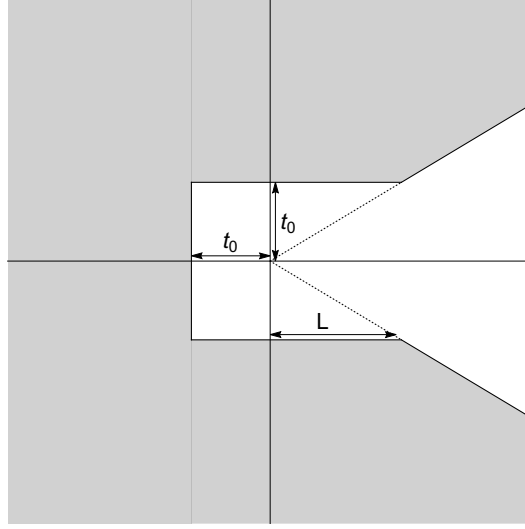


Figure 5.1: The domain $\mathcal{T}_{t_0, L}$.

From Lemma 3.2 we have the existence of a formal solution of the form

$$\mathcal{W}(t) = \begin{pmatrix} \sum_{i \geq 1} W_{x,i} t^{-i} \\ \sum_{i \geq 1} W_{y,i} t^{-i} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{b_0 t} \end{pmatrix} + O(|t|^{-3}),$$

so let

$$\mathcal{W}_N(t) = \begin{pmatrix} \sum_{i=1}^N W_{x,i} t^{-i} \\ \sum_{i=1}^N W_{y,i} t^{-i} \end{pmatrix}.$$

From Taylor's theorem we get the existence of a constant $c'_N > 0$ such that for the partial sum it holds $|\mathcal{W}_N(t) - F(\mathcal{W}_N(t-1))| \leq c'_N |t|^{-N-2}$. To check this it suffices to check that $|\mathcal{W}_1(t) - F(\mathcal{W}_1(t-1))| = O(|t|^{-3})$. Then each extra order in \mathcal{W}_N cancels one order of the remainder.

5.1.1 Approximation of the separatrices

Before stating the result, a few notions need to be defined. Let $L, r \geq 0$ and

$$\text{SQ}(r) := \{z \in \mathbb{C} : |\text{Im}(z)| \leq r, \text{Re}(z) \geq -r\},$$

$$\text{CO}(L, r) := \{z \in \mathbb{C} : \text{Re}(z) \geq 0 \text{ and } t_0 \cdot \text{Re}(z) \geq L \cdot |\text{Im}(z)|\}.$$

Definition 5.1. For any $t_0 > 1$, $L \geq 0$ let $\mathcal{T}_{t_0,L} := \mathbb{C} \setminus (\text{SQ}(t_0)) \cap \text{CO}(L, t_0)$. See Figure 5.1.

Definition 5.2. Let D be an subset of \mathbb{C} with non-empty interior, $X \in \mathbb{C}[[\frac{1}{t}]]$, $N \in \mathbb{N}$ and $C > 0$. Then

$$\mathcal{A}(X, N, D, C) := \{u \in C^\omega(D, \mathbb{C}^2) : |u(t) - X_N(t)| \leq C|t|^{-N-1}\},$$

where X_N we denote the truncation of X at order N .

Notice that if $C_1 \geq C_2$ then $\mathcal{A}(X, N, D, C_2) \subset \mathcal{A}(X, N, D, C_1)$.

Now we can state the theorem about the approximation of the separatrices.

Theorem 5.3. *Let F be as described above and let \mathcal{W} be the formal solution of*

$$W(t+1) = F(W(t)).$$

For $M, N \in \mathbb{N}$, $M, N \geq 2$, $C_M, C_N > 0$ there exists $t_0 \geq 1$ such that for all $w_N \in \mathcal{A}(\mathcal{W}, N, \mathcal{T}_{t_0,L}, C_N)$, all $w_M \in \mathcal{A}(\mathcal{W}, M, \mathcal{T}_{t_0,L}, C_M)$ and all $t \in \mathcal{T}_{t_0,0}$ the following are true.

1. *The limit $W^-(t) := \lim_{m \rightarrow \infty} F_0^m(w_N(t-m))$ exists, is an analytic function on $\mathcal{T}_{t_0,L}$ and*

$$\lim_{m \rightarrow \infty} F_0^m(w_M(t-m)) = \lim_{n \rightarrow \infty} F_0^n(w_N(t-n)).$$

2. $W^-(t) = F_0(W^-(t-1))$.
3. *There exists $C_1 > 0$ such that $\|W^-(t) - w_N(t)\|_\infty \leq C_1|t|^{-N-1}$.*
4. *There exists $C_2 > 0$ such that for all $m \in \mathbb{N}$*

$$\|W^-(t) - F_0^m(w_N(t-m))\|_\infty \leq C_2|t-m|^{-N+1}.$$

5. *There exists $C_3 > 0$ such that for all $m \in \mathbb{N}$*

$$\|\dot{W}^-(t) - (F_0^m)'(w_N(t-m)) \cdot \dot{w}_N(t-m)\|_\infty \leq C_3|t-m|^{-N}.$$

This enables us to calculate values for the stable and the unstable manifolds close to the imaginary axes. This theorem is a corollary of Theorem 5.4 and it will be proved later in the chapter.

5.1.2 Approximation algorithm

The algorithm used to get an approximate value for θ is described in the following steps:

- Fix two integers $N \geq 2$, $m \geq 1$ and a real number $t > 1$.
- Set

$$\begin{aligned} W_{\text{neg}} &\leftarrow \sum_{n=1}^N W_n(-m - \mathfrak{i}t)^{-n}, \\ W_{\text{pos}} &\leftarrow \sum_{n=1}^N W_n(m - \mathfrak{i}t)^{-n}, \\ W_{\text{tan}} &\leftarrow \sum_{n=1}^N -n W_n(-m - \mathfrak{i}t)^{-n-1}. \end{aligned}$$

- Do the following m times

$$\begin{aligned} W_{\text{tan}} &\leftarrow F'(W_{\text{neg}}) \cdot W_{\text{tan}}, \\ W_{\text{neg}} &\leftarrow F(W_{\text{neg}}), \\ W_{\text{pos}} &\leftarrow F^{-1}(W_{\text{pos}}). \end{aligned}$$

- Return the value

$$\theta_{\text{alg}}(N, m, t) = e^{2\pi t} \omega(W_{\text{pos}} - W_{\text{neg}}, W_{\text{tan}}).$$

Let

$$\theta_{\text{app}}(t) = e^{2\pi t} \omega(W^+(t) - W^-(t), \dot{W}^-(t)).$$

We will see below the difference between the separatrix and W_{neg} is of the order $|m + \mathfrak{i}t|^{-N+1}$ and there is a similar error for W_{pos} and W_{tan} . This shows that we have

$$\theta_{\text{app}}(t) = \theta_{\text{alg}}(N, m, t) + O(e^{2\pi t} |m + \mathfrak{i}t|^{-N+1}).$$

Which implies that

$$\theta = \theta_{\text{alg}}(N, m, t) + O(t^5 e^{-2\pi t}) + O(e^{2\pi t} |m + \mathfrak{i}t|^{-N+1}).$$

Of course since we need to compute $\theta_{\text{alg}}(N, m, t)$ numerically, we do not get the true value but rather $\theta_{\text{num}}(N, m, t)$ and finally we get that

$$\theta = \theta_{\text{num}}(N, m, t) + O(t^5 e^{-2\pi t}) + O(e^{2\pi t} |m + it|^{-N+1}) + \text{numerical error}.$$

Using interval numerics we can track the numerical error precisely. However there is no straightforward method that gives the constants that bound the other two terms.

5.2 Approximation theorem

Theorem 5.4. *Let D be an open neighbourhood of the origin in \mathbb{C}^2 , let $F : D \rightarrow \mathbb{C}^2$ be analytic with $F(0) = 0$. Let there exist a formal series $\mathscr{W} \in \frac{1}{t}\mathbb{C}[[\frac{1}{t}]]$ such that $\mathscr{W}(t+1) = F(\mathscr{W}(t))$.*

We assume that for all $N \in \mathbb{N}$, $N \geq 2$ there exists $T_0 > 1$, $C_N > 0$ and $C_J > 0$ such that for all $w \in \mathscr{A}(\mathscr{W}, N, \mathcal{T}_{T_0, L}, C_N)$ and all $t \in \mathcal{T}_{T_0, L}$ it holds

$$\|F'(w(t))\|_{\infty} \leq 1 + 2|t|^{-1} + C_J|t|^{-2}.$$

Then for all $M, N \in \mathbb{N}$, $M, N \geq 2$, $C_M, C_N > 0$ there exists $t_0 \geq T_0$ such that for all $w_N \in \mathscr{A}(\mathscr{W}, N, \mathcal{T}_{t_0, L}, C_N)$, all $w_M \in \mathscr{A}(\mathscr{W}, M, \mathcal{T}_{t_0, L}, C_M)$ and all $t \in \mathcal{T}_{t_0, L}$ the following are true.

1. *The limit $w(t) := \lim_{m \rightarrow \infty} F^m(w_N(t-m))$ exists, is an analytic function on $\mathcal{T}_{t_0, L}$ and*

$$\lim_{m \rightarrow \infty} F^m(w_M(t-m)) = \lim_{n \rightarrow \infty} F^n(w_N(t-n)).$$

2. *$w(t) = F(w(t-1))$.*
3. *There exists $C_1 > 0$ such that $\|w(t) - w_N(t)\|_{\infty} \leq C_1|t|^{-N-1}$.*
4. *There exists $C_2 > 0$ such that for all $m \in \mathbb{N}$*

$$\|w(t) - F^m(w_N(t-m))\|_{\infty} \leq C_2|t-m|^{-N+1}.$$

5. *There exists $C_3 > 0$ such that for all $m \in \mathbb{N}$*

$$\|\dot{w}(t) - (F^m)'(w_N(t-m)) \cdot \dot{w}_N(t-m)\|_{\infty} \leq C_3|t-m|^{-N}.$$

Notice that the analyticity of the limit propagates because of the second property. This implies that if F is entire, $w(t)$ is also an entire function. Also $w(t)$ goes to 0 as t goes to $-\infty$, so it is the unstable manifold. The result for the stable manifold is basically the same but reflected around the imaginary axis because the series is asymptotic to the unstable manifold as t approaches infinity following the negative real line, but it is also asymptotic to the stable manifold as t approaches infinity following the positive real line. So the stable manifold can be treated as the unstable manifold of the inverse map.

Proof of Theorem 5.3. To prove the theorem we just need to check that there exists T_0 such that for all $N \geq 2$, $C > 0$, $r_\rho(t) \in C^\omega(\mathcal{T}_{T_0,L}, \mathbb{C}^2)$ with $|r_\rho(t)| \leq c_\rho |t|^{-3}$ and $w \in \mathcal{A}(\mathcal{W}, N, \mathcal{T}_{T_0,L}, C)$. Then there exists $C_J > 0$ such that

$$\|F'(w(t) + \tau r_\rho(t))\|_\infty \leq 1 + 2|t|^{-1} + C_J |t|^{-2}.$$

We have assumed that F is analytic in a neighbourhood of the origin, so for all $C > 0$ there exists some $T_0 > 0$ such that for all $t \in \mathcal{T}_{T_0,L}$ and all $w \in \mathcal{A}(\mathcal{W}, 2, \mathcal{T}_{T_0,L}, C)$, $\|F(w(t))\|_\infty < \infty$.

Now let $C > 0$ and $w \in \mathcal{A}(2, \mathcal{T}_{T_0,L}, C)$. Let $r_\rho(t) := w(t) - \mathcal{W}_2(t)$. Then $|r_\rho(t)| \leq C|t|^{-3}$. We have

$$F'(w(t)) = F'(\mathcal{W}_2(t) + r_\rho(t)),$$

and

$$F'(z, \zeta) = \begin{pmatrix} 1 - 2b_0y + b_0^2(3x^2 + y^2) & -2b_0x + 2b_0^2xy \\ -2b_0x + 2b_0^2xy & 1 + 2b_0y + b_0^2(x^2 + 3y^2) \end{pmatrix} + R_J(x, y),$$

with

$$\|R_J(x, y)\|_\infty \leq C'_J \|(x, y)\|_\infty^3.$$

So we get

$$F'(\mathcal{W}_2(t) + r_\rho(t)) = \begin{pmatrix} 1 - 2t^{-1} + t^{-2} & 0 \\ 0 & 1 + 2t^{-1} + 3t^{-2} \end{pmatrix} + R_J(t)$$

with

$$|R_J(t)| \leq C_J |t|^{-3}.$$

By Lemma 4.9 we have the result.

If $N > 2$ then let $w \in \mathcal{A}(\mathcal{W}, N, \mathcal{T}_{T_0, L}, C)$. There exists $C' > 0$ such that $w \in \mathcal{A}(2, \mathcal{T}_{T_0, L}, C')$, so we are in the previous situation. \square

In order to prove Theorem 5.4, we need a few intermediate results.

Lemma 5.5. *Let $t_0 > 1$, $t \in \mathcal{T}_{t_0, L}$ and $N \geq 2$. Then there exists $C_\lambda \geq 1$ such that*

$$\sum_{j=0}^m |t - j|^{-N} \leq C_\lambda |t|^{-N+1}.$$

Proof. We have

$$\sum_{j=1}^m |t - j|^{-N} \leq \int_0^m |t - x|^{-N} dx \leq \int_0^\infty |t - x|^{-N} dx.$$

To bound the integral we consider two cases.

- $\operatorname{Re}(t) < 0$

Let $t = -r + is$. Then

$$\int_0^\infty |t - x|^{-N} dx = \int_0^\infty ((r + x)^2 + s^2)^{-N/2} dx.$$

It holds

$$\begin{aligned} \int_0^\infty ((r + x)^2 + s^2)^{-N/2} dx &\leq \int_0^\infty (r + x)^{-N} dx \leq \frac{r^{-N+1}}{N-1} \leq \frac{\pi}{2} r^{-N+1} \\ \Rightarrow r^{N-1} \int_0^\infty ((r + x)^2 + s^2)^{-N/2} dx &\leq \frac{\pi}{2}. \end{aligned}$$

Also

$$\int_0^\infty ((r + x)^2 + s^2)^{-N/2} dx \leq \int_0^\infty (x^2 + s^2)^{-N/2} dx \leq \int_0^{\frac{\pi}{2}} \frac{|s| \sec^2(x)}{|s|^N \sec^N(x)} dx$$

$$\begin{aligned}
&= |s|^{-N+1} \int_0^{\frac{\pi}{2}} \cos^{N-2}(t) dx = |s|^{-N+1} \frac{\sqrt{\pi} \Gamma(\frac{N-1}{2})}{2\Gamma(\frac{n}{2})} \\
&\leq \frac{\pi}{2} |s|^{-N+1}
\end{aligned}$$

thus

$$|s|^{N-1} \int_0^\infty ((t+x)^2 + s^2)^{-N/2} dx \leq \frac{\pi}{2}.$$

Let $|\cdot|_n$ denote the n -norm on the plane. We get

$$(r^{N-1} + |s|^{N-1}) \int_0^\infty ((r+x)^2 + s^2)^{-N/2} dx \leq \pi$$

so

$$\int_0^\infty ((r+x)^2 + s^2)^{-N/2} dx \leq \pi |t|_{N-1}^{-N+1}$$

Since all norms on the plane are equivalent, for each $N \geq 2$, there exists a constant $C_2 > 0$ such that

$$\int_0^\infty |t-x|^{-N} dx \leq \pi C_2 |t|_2^{-N+1}.$$

- $\operatorname{Re}(t) \geq 0$

Let $t = r + \mathbf{i}s$. It holds

$$\begin{aligned}
\int_0^\infty ((x-r)^2 + s^2)^{-N/2} dx &\leq \int_{-\infty}^\infty ((x-r)^2 + s^2)^{-N/2} dx \\
&\leq \int_{-\infty}^\infty (x^2 + s^2)^{-N/2} dx \\
&\leq 2 \int_0^\infty (x^2 + s^2)^{-N/2} dx \leq \pi |s|^{-N+1}.
\end{aligned}$$

By construction we have $0 < \arg(L + \mathbf{i}t_0) \leq |\arg(t)| \leq \pi/2$, so $|s| \geq |t| \cos(L + \mathbf{i}t_0)$. Thus

$$\int_0^\infty ((x-r)^2 + s^2)^{-N/2} dx \leq \pi (\cos(\arg(L + \mathbf{i}t_0)))^{-N+1} |t|^{-N+1} = C'_2 |t|^{-N+1}.$$

Let $C = \max\{\pi \cdot C_2, \pi \cdot (\cos(\arg(L + it_0)))^{-N+1}\}$. Then

$$\begin{aligned}
\sum_{j=0}^m |t-j|^{-N} &\leq |t|^{-N} + \int_0^\infty |t-x|^{-N} dx \\
&\leq |t|^{-N} + C|t|^{-N+1} \\
&\leq (1+C)|t|^{-N+1} \\
&= C_\lambda |t|^{-N+1}. \quad \square
\end{aligned}$$

Lemma 5.6. *For any $t_0 > 1$ and any $L > 0$ if $t + m \in \mathcal{T}_{t_0, L}$ with $|\operatorname{Re}(t)| \leq 1$ and $m \in \mathbb{N}$, then there exists $C_L > 0$ such that*

$$\frac{1}{|t|} \leq C_L \frac{1}{|t+m|}.$$

Proof. For all $t \in \mathcal{T}_{t_0, L}$ it is true that $t_0 \cdot \operatorname{Re}(z) < L \cdot |\operatorname{Im}(z)|$. From this the result follows. \square

To prove Theorem 5.4 we need to check that the error does not grow too quickly with the number of iterations. The next lemma gives a bound on the error after m steps. For this we use the bound for the Jacobian since this is the worst case scenario of the error propagation of one step.

Lemma 5.7. *Let $\mu : \mathbb{C}^* \rightarrow \mathbb{R}_+$ with $\mu(t) \leq 1 + 2|t|^{-1} + C_\mu|t|^{-2}$ and $P_m^-(t) := \prod_{k=0}^m \mu(t-k)$ with $m \in \mathbb{N}$. Then there exists $C_\rho > 0$ such that $\forall t \in \mathcal{T}_{t_0, 0}$ it holds*

$$P_m^-(t) \leq C_\rho \frac{|t-m|^2}{|t|^2}$$

with

$$C_\rho = \left(1 + \frac{2}{t_0} + \frac{C_\mu}{t_0^2}\right) \cdot \exp\left(2\pi + \frac{\pi}{t_0} \left(C_\mu + \left(4 + \frac{C_\mu}{t_0}\right)^2\right)\right).$$

Proof. The same as the proof of Lemma 4.12. \square

Notice that given C_μ , there exists t_0 such that C_ρ can be arbitrarily close to $e^{2\pi}$.

Lemma 5.8. *Let $\mu : \mathbb{C}^* \rightarrow \mathbb{R}_+$ with $\mu(t) \leq 1 + 2|t|^{-1} + C_\mu|t|^{-2}$ and $P_m^+(t) :=$*

$\prod_{k=0}^m \mu(t+k)$ with $m \in \mathbb{N}$. Then there exists $C_\rho > 0$ such that $\forall t \in -\mathcal{T}_{t_0,0}$ it holds

$$P_m^+(t) \leq C_\rho \frac{|t+m|^2}{|t|^2}$$

with

$$C_\rho = \left(1 + \frac{2}{t_0} + \frac{C_\mu}{t_0^2}\right) \cdot \exp\left(2\pi + \frac{\pi}{t_0} \left(C_\mu + \left(4 + \frac{C_\mu}{t_0}\right)^2\right)\right).$$

Proof. The same as above but reflected by the imaginary axis. \square

The next lemma states that if the initial point is close enough to the fixed point, then m iterations do not move that point “too far” from the separatrix.

Lemma 5.9. *For $N \geq 2$ and $C_N > 0$ there exists $t_0 > 1$ and $C_1 > 0$ such that for all $w \in \mathcal{A}(N, \mathcal{T}_{t_0,L}, c_N)$, all $m \in \mathbb{N}$ and all $t \in \mathcal{T}_{t_0,L}$ it is true that*

$$\|w(t) - F^m(w(t-m))\|_\infty \leq C_1 |t|^{-N-1}. \quad (5.1)$$

Proof. Choose $t \in \mathcal{T}_{t_0,0}$ and let

$$r_{n,N}(t) := w(t) - F^n(w(t-n))$$

and

$$I_{n,N}(t) := \int_0^1 F'(w(t) + \tau r_{n,N}(t)) d\tau$$

with $r_{0,N}(t) = 0$. We have

$$\begin{aligned} r_{n,N}(t) &= w(t) - F^n(w(t-n)) \\ &= F(w(t-1)) + r_{1,N}(t) - F(w(t-1) - r_{n-1,N}(t-1)) \\ &= I_{n-1,N}(t-1) \cdot r_{n-1,N}(t-1) + r_{1,N}(t). \end{aligned}$$

Fix $m \in \mathbb{N}$. Then for $1 \leq n \leq m$ we have

$$\begin{aligned} r_{n,N}(t-m+n) &= I_{n-1,N}(t-m+n-1) \cdot r_{n-1,N}(t-m+n-1) \\ &\quad + r_{1,N}(t-m+n). \end{aligned}$$

We define

$$\begin{aligned}\delta_n &= \|r_{n,N}(t - m + n)\|_\infty, \\ \alpha_n &= \|I_{n,N}(t - m + n)\|_\infty, \\ \beta_n &= \|r_{1,N}(t - m + n + 1)\|_\infty.\end{aligned}$$

So for $1 \leq n \leq m$ we have

$$\delta_n \leq \alpha_{n-1} \delta_{n-1} + \beta_{n-1},$$

which gives

$$\delta_m \leq \sum_{n=0}^{m-1} \left(\prod_{k=n+1}^{m-1} \alpha_k \right) \beta_n.$$

By Taylor's theorem we have $|r_{1,N}(t)| \leq C_N |t|^{-N-1}$. Set $C'_1 = (1 + \frac{2}{t_0} + \frac{C_N}{t_0^2}) e^{2\pi+1} C_\lambda C_N$. Assume that for all $1 \leq n < m$ we have $|r_{n,N}(t)| \leq C'_1 |t|^{-N-1}$. Then by assumption there exists a constant $C_J > 0$ depending on C'_1 , such that $\alpha_n \leq 1 - 2|t - m + n|^{-1} + C_J |t - m + n|^{-2}$.

This gives

$$\begin{aligned}\delta_m &\leq \sum_{n=0}^{m-1} \left(\prod_{k=n+1}^{m-1} 1 - 2|t - m + k|^{-1} + C_J |t - m + k|^{-2} \right) \cdot C_N |t - m + n + 1|^{-N-2} \\ &= \sum_{n=0}^{m-1} \left(\prod_{j=1}^{m-n-1} 1 - 2|t - j|^{-1} + C_J |t - j|^{-2} \right) \cdot C_N |t - m + n + 1|^{-N-2} \\ &= \sum_{n=0}^{m-1} P_{1,m-n-1}^-(t) \cdot C_N |t - m + n + 1|^{-N-2} \\ &\leq \sum_{n=0}^{m-1} C_\rho \frac{|t - m + n + 1|^2}{|t|^2} \cdot C_N |t - m + n + 1|^{-N-2} \\ &\leq C_\rho C_N |t|^{-2} \sum_{n=0}^{m-1} |t - m + n + 1|^{-N} \\ &\leq C_\rho C_N |t|^{-2} \sum_{j=0}^{m-1} |t - j|^{-N} \\ &\leq C_\rho C_N |t|^{-2} C_\lambda |t|^{-N+1} \\ &= C_\rho C_\lambda C_N |t|^{-N-1}.\end{aligned}$$

From Lemma 5.7 we have that $C_\rho = \exp(2\pi + \gamma(C_1, t_0)/t_0)$ with γ decreasing with t_0 . Then we choose t_0 such that

$$C_\rho = \exp(2\pi + \gamma(C'_1, t_0)/t_0) \leq \frac{C'_1}{(1 + \frac{2}{t_0} + \frac{C_N}{t_0^2})C_\lambda C_N} = e^{2\pi+1},$$

which translates to

$$\gamma(C'_1, t_0) \leq t_0.$$

By choosing t_0 big enough the above relation becomes true and this implies that $C_\rho C_\lambda C_N \leq C'_1$ and finally $|r_m(t)| \leq C'_1 |t|^{-N-1}$. Then by induction with the same t_0 we get

$$\|r_m(t)\|_\infty \leq C'_1 |t|^{-N-1}$$

for all $m \in \mathbb{N}$.

Now we choose $t \in \mathcal{T}_{t_0, L}$ such that $0 \leq \operatorname{Re}(t) - \kappa \leq 1$ for some $\kappa \in \mathbb{N}$. We repeat the above construction for $m + \kappa$ steps and we end up with the equation

$$\delta_{m+\kappa} \leq \sum_{n=0}^{m+\kappa-1} \left(\prod_{k=n+1}^{m+\kappa-1} \alpha_k \right) \beta_n.$$

As above by assuming that for all $n < m + \kappa$ it holds $|\delta_n(t)| \leq C_1 |t|^{-N-1}$ for a big enough C_1 we get

$$\begin{aligned} \delta_{m+\kappa} &\leq \\ &\leq \sum_{n=0}^{m+\kappa-1} \left(\prod_{k=n+1}^{m+\kappa-1} 1 + \frac{2}{|t-m-\kappa+k|} + \frac{C_J}{|t-m-\kappa+k|^2} \right) \cdot \frac{C_N}{|t-m-\kappa+n+1|^{N+2}} \\ &= \sum_{n=0}^{m-2} \left(\prod_{j=1}^{\kappa} \prod_{j=\kappa+1}^{m+\kappa-n-1} 1 + \frac{2}{|t-j|} + \frac{C_J}{|t-j|^2} \right) \cdot \frac{C_N}{|t-m-\kappa+n+1|^{N+2}} \\ &+ \sum_{n=m-1}^{m+\kappa-1} \left(\prod_{j=1}^{m+\kappa-n-1} 1 + \frac{2}{|t-j|} + \frac{C_J}{|t-j|^2} \right) \cdot \frac{C_N}{|t-m-\kappa+n+1|^{N+2}} \\ &= \sum_{n=0}^{m-2} P_\kappa^+(t-\kappa) \cdot P_{m-n-2}^-(t-\kappa-1) \cdot C_N |t-m-\kappa+n+1|^{-N-2} \\ &+ \sum_{n=m-1}^{m+\kappa-1} P_{m+\kappa-n-1}^+(t-m-\kappa+n+1) \cdot C_N |t-m-\kappa+n+1|^{-N-2} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{m-2} C_\rho \frac{|t|^2}{|t-\kappa|^2} \cdot C_\rho \frac{|t-m-\kappa+n+1|^2}{|t-\kappa-1|^2} \cdot C_N |t-m-\kappa+n+1|^{-N-2} \\
&+ \sum_{n=m-1}^{m+\kappa-1} C_\rho \frac{|t|^2}{|t-m-\kappa+n+1|^2} \cdot C_N |t-m-\kappa+n+1|^{-N-2} \\
&= C_\rho^2 C_N \frac{|t|^2}{|t-\kappa|^2 |t-\kappa-1|^2} \sum_{n=0}^{m-2} |t-m-\kappa+n+1|^{-N} \\
&+ C_\rho C_N |t|^2 \sum_{n=m-1}^{m+\kappa-1} |t-m-\kappa+n+1|^{-N-4} \\
&\leq C_\rho^2 C_N C_\lambda |t|^2 |t-\kappa|^{-2} |t-\kappa-1|^{-N-1} + C_\rho C_N C_\lambda |t|^2 |t-\kappa|^{-N-3}
\end{aligned}$$

Since $0 \leq \operatorname{Re}(t-\kappa) \leq 1$ we can use Lemma 5.6 and by increasing t_0 if necessary we get the result. \square

Now we know enough to find a bound on the Jacobian of F^m .

Lemma 5.10. *For $N \geq 2$, $C_N > 0$ and $r(t) \in C^\omega(\mathcal{T}_{t_0,L}, \mathbb{C}^2)$ with $|r(t)| \leq c|t|^{-3}$, there exists $t_0 > 1$ and $C_P > 0$ such that for all $w_N \in \mathcal{A}(N, \mathcal{T}_{t_0,L}, C_N)$ and all $m \in \mathbb{N}$ it holds*

$$\left\| \int_0^1 (F^m)'(w_N(t-m) + \tau r(t-m)) d\tau \right\|_\infty \leq C_P |t-m|^2.$$

Proof. We have

$$(F^m)'((x, y)) = \prod_{j=1}^m F'(F^{m-j}((x, y))).$$

Since $w_N \in \mathcal{A}(N, \mathcal{T}_{t_0,L}, C_N)$ and $|r(t)| \leq c|t|^{-3}$, we know that for $0 \leq \tau \leq 1$ there exists $c_N > 0$ such that $w_N(t) + \tau r(t) \in \mathcal{A}(2, \mathcal{T}_{t_0,L}, c_N)$. We choose t_0 such that we can use Lemma 5.9 and we get that there exists $c' > 0$ such that for any $n < m \in \mathbb{N}$ it holds

$$F^n(w_N(t-m) + \tau r(t-m)) = w_N(t-m+n) + r'_n(t-m+n)$$

with $|r'_n(t)| \leq c'|t|^{-3}$. So for $\mathcal{T}_{t_0,0}$ we have

$$\|(F^m)'(w_N(t-m) + \tau r(t-m))\|_\infty = \left\| \prod_{j=1}^m F'(F^{m-j}(w_N(t-m) + \tau r(t-m))) \right\|_\infty$$

$$\begin{aligned}
&= \left\| \prod_{j=1}^m F'((w_N(t-j) + r'_j(t-j))) \right\|_{\infty} \\
&\leq \prod_{j=1}^m \|F'((w_N(t-j) + r'_j(t-j)))\|_{\infty} \\
&\leq \prod_{j=1}^m \left(1 + \frac{2}{|t-j|} + C_J|t-j|^{-2}\right)
\end{aligned}$$

Now we use Lemma 5.7 and depending on the sign of $\operatorname{Re}(t)$ we may use also Lemma 5.8 and with Lemma 5.6 we get the result. \square

Proof of Theorem 5.4.

1. For $m, n_1, n_2 \in \mathbb{N}$, $m \geq L$ and $t \in \mathcal{T}_{t_0, L}$ we have $F^{n_i}(w_N(t - n_i)) = w_N(t) + r_{n_i}(t)$ with $\|r_{n_i}(t)\|_{\infty} \leq C|t|^{-N-1}$ by Lemma 5.9, $i \in \{1, 2\}$. Let $r_{n_1, n_2}(t) = r_{n_1}(t) - r_{n_2}(t)$. Then $\|r_{n_1, n_2}(t)\|_{\infty} \leq 2C|t|^{-N-1}$. So

$$\begin{aligned}
&\|F^{m+n_1}(w_N(t - m - n_1)) - F^{m+n_2}(w_N(t - m - n_2))\|_{\infty} \\
&= \|F^m(w_N(t - m) + r_{n_1}(t - m)) - F^m(w_N(t - m) + r_{n_2}(t - m))\|_{\infty} \\
&= \|F^m(w_N(t - m) + r_{n_2}(t - m) + r_{n_1}(t - m) - r_{n_2}(t - m)) \\
&\quad - F^m(w_N(t - m) + r_{n_2}(t - m))\|_{\infty} \\
&= \left\| \left(\int_0^1 (F^m)'(w_N(t - m) + r_{n_2}(t - m) + \tau r_{n_1, n_2}(t - m)) d\tau \right) \cdot r_{n_1, n_2}(t - m) \right\|_{\infty} \\
&\leq \left\| \int_0^1 (F^m)'(w_N(t - m) + r_{n_2}(t - m) + \tau r_{n_1, n_2}(t - m)) d\tau \right\|_{\infty} \cdot \|r_{n_1, n_2}(t - m)\|_{\infty} \\
&\leq C_P |t - m|^2 2C |t - m|^{-N-1} = 2C_P C |t - m|^{-N+1} \leq 2C_P C |t - m|^{-N+1}.
\end{aligned}$$

This shows that the sequence of analytic functions $\{F^m(w_N(t - m))\}_{m \geq 0}$ on $\mathcal{T}_{t_0, L}$ is uniformly Cauchy, which implies that the limit exists and is an analytic function.

Let $M > N \geq 2$. Then $w_M(t) - w_N(t) = R_{M, N}(t)$ with $\|R_{M, N}(t)\|_{\infty} \leq C_{M, N} |t|^{-N-1}$ for some $C_{M, N} > 0$. We have

$$\begin{aligned}
&\|F^m(w_M(t - m)) - F^m(w_N(t - m))\|_{\infty} = \\
&= \|F^m(w_N(t - m) + R_{M, N}(t - m)) - F^m(w_N(t - m))\|_{\infty} \\
&= \left\| \left(\int_0^1 (F^m)'(w_N(t - m) + \tau R_{M, N}(t - m)) d\tau \right) \cdot R_{M, N}(t - m) \right\|_{\infty}
\end{aligned}$$

$$\begin{aligned} &\leq \left\| \int_0^1 (F^m)'(w_N(t-m) + \tau R_{M,N}(t-m)) d\tau \right\|_\infty \cdot \|R_{M,N}(t-m)\|_\infty \\ &\leq C_P |t-m|^2 C_{M,N} |t-m|^{-N-1} \leq C_P C_{M,N} |t-m|^{-N+1}. \end{aligned}$$

Then the limit $m \rightarrow \infty$ yields the desired result.

2. We have

$$\begin{aligned} &\|F^m(w_N(t-m)) - F^{m+1}(w_N(t-m-1))\|_\infty = \\ &= \|F^m(w_N(t-m)) - F^m(w_N(t-m) + r_N(t-m))\|_\infty \\ &\leq \left\| \int_0^1 (F^m)'(w_N(t-m) + \tau r_N(t-m)) d\tau \right\|_\infty \cdot |r_N(t-m)| \\ &\leq C_P |t-m|^2 C_N |t-m|^{-N-2} \leq C_P C_N |t-m|^{-N}. \end{aligned}$$

The limit yields $|w(t) - F(w(t-1))| = 0$.

3. It follows trivially by taking the limit of the Lemma 5.9.

4. We denote $w(t) - w_N(t) = R_N(t)$ so we have

$$\begin{aligned} &\|w(t) - F^m(w_N(t-m))\|_\infty = \|F^m(w(t-m)) - F^m(w_N(t-m))\|_\infty \\ &= \|F^m(w(t-m)) - F^m(w(t-m) - R_N(t-m))\|_\infty \\ &\leq \left\| \int_0^1 (F^m)'(w(t-m) + \tau R_N(t-m)) d\tau \right\|_\infty \cdot \|R_N(t-m)\|_\infty \\ &\leq C_P |t-m|^2 C_1 |t-m|^{-N-1} = C_P C_1 |t-m|^{-N+1}. \end{aligned}$$

5. We differentiate the relation $w(t) - F^m(w_N(t-m)) = R_{N,m}(t)$ to get

$$\dot{w}(t) - (F^m)'(w_N(t-m)) \cdot \dot{w}_N(t-m) = \dot{R}_{N,m}(t).$$

Then we use the bound of $R_{N,m}$ to get the result. □

5.3 The Stokes constant of the Hénon map

As an example for the computation of the Stokes constant we will use the map

$$h : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto R_{2\pi/3} \cdot \begin{pmatrix} x \\ y - x^2 \end{pmatrix},$$

which is an instance of the Hénon map and it is at 1:3 resonance. We set $H = h^3$ and we have the following theorem.

Proposition 5.11. *There exists a unique formal solution that satisfies $\mathcal{W}(t+1) = H(\mathcal{W}(t))$ and has the form*

$$\mathcal{W}(t) = \begin{pmatrix} \frac{1}{\sqrt{3}}t^{-2} \\ -\frac{4}{3}t^{-1} + \frac{7}{13}t^{-3} \end{pmatrix} + O(t^{-4}).$$

Moreover Theorem 5.4 can be applied on H .

Proof. Straightforward computations. □

Theorem 5.11 implies that we can use the algorithm described in Section 5.1.2 to calculate the Stokes constant of H . Notice that even though the rate of convergence is known, the constants involved are not and this prohibits the construction of a computer assisted proof. Still a numerical experiment can give a good approximation.

Numerical results

Because of the symmetry of the map it is expected that the Stokes constant is either real or imaginary. As we will see in this case the Stokes constant appears to be imaginary.

The program for the computation was written in Julia language and the numerical error was tracked using the interval arithmetics library ValidatedNumerics.¹

For the calculation we fix $N = 200$ and $t = 50$, m will be varied from 500 to 10000 with step 500, we have $t^5 e^{-2\pi t} \approx 1.14 \times 10^{-128}$. Since we do not know the constant that bounds the approximation error we use the constant of the next term and we get $|W_{201}|e^{2\pi t}|M - it|^{1-N} \approx 4.19 \times 10^{-442}$. Since N and t are fixed, instead of writing $\theta_{\text{num}}(N, m, t)$, we will write $\theta_{\text{num}}(m)$. We also write M for the maximum value of m , i.e. 10000.

We get

$$\text{Re}(\theta_{\text{num}}(M)) = -1.2861332396 \dots \times 10^{-127}$$

¹ <https://github.com/dpsanders/ValidatedNumerics.jl>

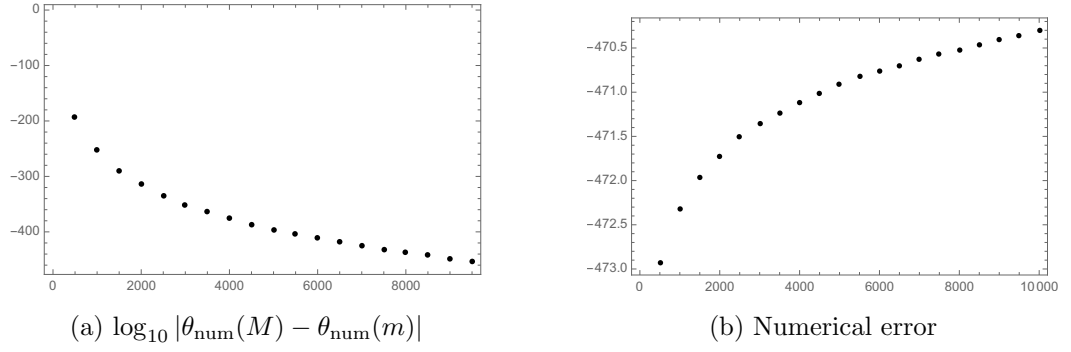


Figure 5.2: Case $t = 50$: Figure (a) shows the rate of convergence to $\theta_{\text{num}}(M)$. Figure (b) shows the base 10 logarithm of the numerical error. In both figures the horizontal axis is the number of iterations.

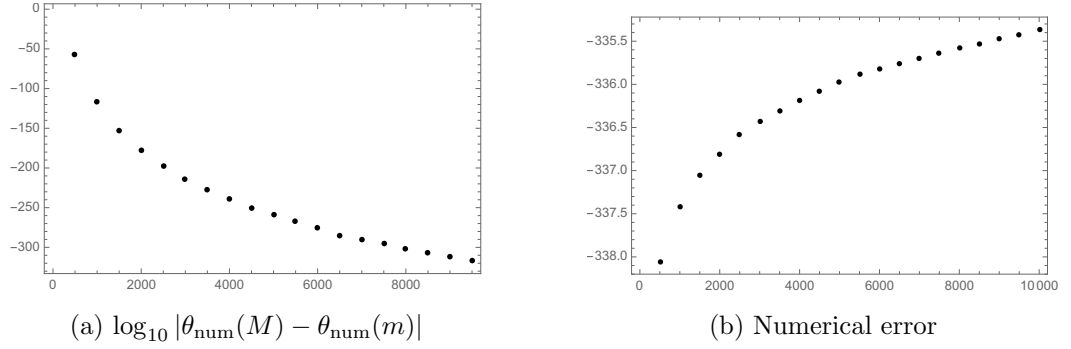


Figure 5.3: Case $t = 100$: Figure (a) shows the rate of convergence to $\theta_{\text{num}}(M)$. Figure (b) shows the base 10 logarithm of the numerical error. In both figures the horizontal axis is the number of iterations.

$$\text{Im}(\theta_{\text{num}}(M)) = 7247.74134408 \dots$$

We see that the calculation converges quickly as m grows. However the dominant error term is the one that appears by ignoring the terms of order $t^5 e^{-2\pi t}$ in the splitting. Since this has size 10^{-128} we should not trust more than 127 digits of the computation.

Then we fix $N = 200$ and $t = 100$, with m varying from 500 to 10000 with step 500. In this case $t^5 e^{-2\pi t} \approx 1.33 \times 10^{-263}$ and $|W_{201}| e^{2\pi t} |M - it|^{1-N} \approx 1.13 \times 10^{-305}$. We choose again M to be equal to 10000.

We get

$$\begin{aligned} \text{Re}(\theta_{\text{num}}(M)) &= -1.8908405405 \dots \times 10^{-263} \\ \text{Im}(\theta_{\text{num}}(M)) &= 7247.74134408 \dots \end{aligned}$$

with the 127 first digits of $\text{Im}(\theta_{\text{num}}(M))$ coinciding with the calculation at $t = 50$. The dominant error term is the same as before but this time its size is 10^{-263} . This means that we can trust no more than 262 digits of the result.

Since we expect the Stokes constant to be imaginary, the real part of the computation serves as an extra method to assess the error. We see that it agrees with what we expected from the analysis. The above serve as strong indication that for the Hénon map, the constant θ does not vanish.

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